Assignment 0: Newmark time-stepping

- In Assignment A0, tasks 4.2 - 4.9 you are asked to implement a Newmark scheme for time stepping.
- We know
  1. the differential equation of motion of the system we want to simulate (2nd order in time).
     Note: This equation tells us how the "acceleration" $\ddot{\underline{u}}(t)$ behaves. From the Galerkin discretization:
     $$\underline{\underline{M}} \cdot \ddot{\underline{u}}(t) = -\underline{\underline{K}} \cdot \underline{u}(t) \tag{1}$$
  2. the state of the system at time t_i, i.e., the value of the unknown $\underline{u}_i$ and the "velocity" $\dot{\underline{u}}_i$.
- We want to
  → find a *good* way of computing the unknown state of the system at the next time step t_{i+1}.
- Problem: Equation (1) gives us one equation to find $\ddot{\underline{u}}_{i+1} = \ddot{\underline{u}}(t_{i+1})$. However, together with $\dot{\underline{u}}_{i+1}$ and $\underline{u}_{i+1}$ we have a total of three unknowns! → We need two more equations.

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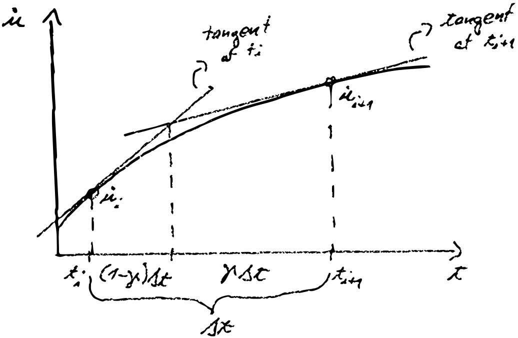
Mean value ansatz for the velocity

- The Newmark scheme is based on the idea, that we can get the correct value of $\dot{\underline{u}}$ at the new time t_{i+1} by using a combination of the slopes at the current time t_i (i.e., $\ddot{\underline{u}}_i$) and the new time t_{i+1} (i.e., $\ddot{\underline{u}}_{i+1}$). See the following sketch:



- We can obtain the new velocity $\dot{\underline{u}}_{i+1}$ as

$$\dot{\underline{u}}_{i+1} = \dot{\underline{u}}_i + (1-\gamma)\Delta t \ddot{\underline{u}}_i + \gamma \Delta t \ddot{\underline{u}}_{i+1} \,. \qquad (2)$$

Idea of the Newmark scheme

- Assume we know $\dot{\underline{u}}_i$. (We do because we know the state at time t_i).
- Equation (2) yields the exact value for the unknown $\dot{\underline{u}}_{i+1}$ if we pick the value of $\gamma \in [0, 1]$ correctly.
- We don't know the correct value for γ, but we can make a guess to end up having a better approximation to $\dot{\underline{u}}_{i+1}$ than simply using the forward scheme $\dot{\underline{u}}_{i+1} = \dot{\underline{u}}_i + \Delta t \ddot{\underline{u}}_i$.
- But how do we get the unknown $\underline{u}_{i+1}$? $\rightarrow$ We need one more equation.

Idea of the Newmark scheme II

- Similarly, we can use a mean-value ansatz to obtain the new values $\underline{u}_{i+1}$. However, we use a second order Taylor series for $\underline{u}(t)$:

$$\underline{u}_{i+1} = \underline{u}_i + \Delta t \dot{\underline{u}}_i + \frac{1}{2}\Delta t^2 \ddot{\underline{u}}_\beta \,, \tag{3}$$

with

$$\ddot{\underline{u}}_\beta = (1-2\beta)\ddot{\underline{u}}_i + 2\beta\ddot{\underline{u}}_{i+1} \,, \quad \beta \in [0, 1/2] \,. \tag{4}$$

- Combining the last two equations yields

$$\underline{u}_{i+1} = \underline{u}_i + \Delta t \dot{\underline{u}}_i + \frac{1}{2}\Delta t^2 \left[(1-2\beta)\ddot{\underline{u}}_i + 2\beta\ddot{\underline{u}}_{i+1}\right] \,, \tag{5}$$

- Again, due to the mean-value theorem, if we choose the correct value for β, we end up with the exact value of $\underline{u}_{i+1}$.
- But we don't know β, so we also make a guess for the value of β – which will hopefully give us a good approximation for $\underline{u}_{i+1}$.

Remark on the Newmark parameters

Note: You can choose the value of the Newmark parameters γ and β as you desire (within their definition range). If their values are arbitrary, how can you still end up with correct solutions of your simulation?

- Because as long as Δt is small enough, it does not matter whether you use
  - $\rightarrow$ $\dot{\underline{u}}_i$ or $\dot{\underline{u}}_{i+1}$ (see sketch)
  - $\rightarrow$ $\ddot{\underline{u}}_i$ or $\ddot{\underline{u}}_{i+1}$.

  (Requires the solutions to be smooth.)
- Why are γ and β there then in the first place? Because depending on your choice of the Newmark parameters, you might get better or worse results.
  - $\rightarrow$ Most importantly, it affects the stability conditions of the time-stepping scheme (restrictions on Δt).
    Note: if your time-stepping is unstable, your errors may grow over time without bound (errors $\rightarrow \infty$). You don't want that!
- Questions:
  - $\rightarrow$ For what values of γ, β is the time-stepping unconditionally stable? Does that mean that you don't have errors?
  - $\rightarrow$ What is an explicit and an implicit scheme? What is the difference between the two?

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The Newmark time-stepping

- From (1) we know that

$$\underline{\underline{M}} \cdot \underline{\ddot{u}}_{i+1} = -\underline{\underline{K}} \cdot \underline{u}_{i+1} \,. \tag{6}$$

- With (6), (2) and (5) we now have three equations for the three unknowns $\underline{\ddot{u}}_{i+1}$, $\underline{\dot{u}}_{i+1}$ and $\underline{u}_{i+1}$. → We can solve the system.
- **Task**: Insert (5) into (6) and solve for $\underline{\ddot{u}}_{i+1}$. You can use the result to
  1. Compute $\underline{\ddot{u}}_{i+1}$.
  2. Knowing $\underline{\ddot{u}}_{i+1}$, you can compute $\underline{\dot{u}}_{i+1}$ with (2).
  3. Knowing $\underline{\dot{u}}_{i+1}$ and $\underline{\ddot{u}}_{i+1}$, you can compute $\underline{u}_{i+1}$ using (5).
  4. Knowing $\underline{u}_{i+1}$, $\underline{\dot{u}}_{i+1}$ and $\underline{\ddot{u}}_{i+1}$, you can compute $\underline{u}_{i+2}$, $\underline{\dot{u}}_{i+2}$ and $\underline{\ddot{u}}_{i+2}$! :)

  → In other words: You can perform a transient simulation of your system!

References

Have fun simulating!

- For the Wikipedia article, refer to
  → https://en.wikipedia.org/wiki/Newmark-beta_method.
- The idea of the method is based on the *extended mean value theorem*, see
  → https://en.wikipedia.org/wiki/Mean_value_theorem#Cauchy.27s_mean_value_theorem.

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