Guitar string problem

- boundary value problem (BVP):

$$
\begin{aligned}
Tu''(x,t) &= \mu \ddot{u}(x,t) \quad \text{on} \quad x \in \Omega = [0, L]\,, && \text{(1a)}\\
u(0,t) &= 0\,, && \text{(1b)}\\
u(L,t) &= 0\,. && \text{(1c)}
\end{aligned}
$$

- weak form:

$$
-T \int_0^L v'(x) u'(x,t)\,\mathrm{d}x = \frac{\partial^2}{\partial t^2} \mu \int_0^L v(x) u(x,t)\,\mathrm{d}x \qquad (2)
$$

- semidiscrete Galerkin formulation:

$$
-\underline{\underline{K}} \cdot \underline{u}(t) = \frac{\partial^2}{\partial t^2} \underline{\underline{M}} \cdot \underline{u}(t) \qquad (3)
$$

Corresponding Eigenvalue Problem

- harmonic ansatz in time:

$$\underline{u}(t) = \underline{\hat{u}}\,\mathrm{e}^{\mathrm{i}\omega t} \quad \Rightarrow \quad \underline{\ddot{u}}(t) = (\mathrm{i}\omega)^2 \underline{\hat{u}}\,\mathrm{e}^{\mathrm{i}\omega t} \tag{4}$$

- generalized eigenvalue problem:

$$\underline{\underline{K}} \cdot \underline{\hat{u}} = \lambda\, \underline{\underline{M}} \cdot \underline{\hat{u}} \tag{5}$$

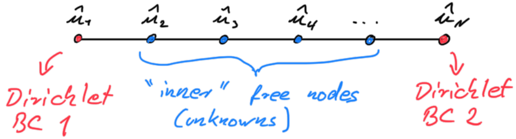
with

$$\lambda = \omega^2 \,. \tag{6}$$

- We would like to solve for all eigenvalues λ_p and corresponding eigenvectors $\underline{\hat{u}}_p$. The eigenvectors are also known as the *mode structure*.

Incorporate Dirichlet BC at $n = 1$

- mesh:



- For this example assume $N = 3$, i.e., a 3×3 system.
- Consider the node $n = 1$ to be a Dirichlet node with boundary condition:

$$\hat{u}_1 = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \cdot \underline{\hat{u}} = \lambda \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \underline{\hat{u}} \tag{7}$$

- Replace into the corresponding equation of the full eigenvalue problem:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \lambda \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$$

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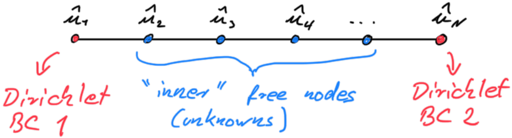

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Continuation: Incorporate Dirichlet BCs

- Additionally: as $\hat{u}_1 = 0$, the values K_{i1} and $M_{i,1}$, where the index i corresponds to **free nodes** (not Dirichlet nodes), are irrelevant. We can, therefore, set them to zero:

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- The previous procedure will ensure the desired condition $\hat{u}_1 = 0$.
- Proceed analogously for any other Dirichlet nodes you may have in your problem.
- **Question:** Could we set the "one entry" in the stiffness matrix instead of the mass matrix? If so, what would be the problem regarding the transformation of the generalized eigenvalue problem to an ordinary eigenvalue problem?

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