Handling Dirichlet Boundary Conditions

- assume a 1D-problem with Dirichlet Boundary Conditions (BCs) on both edges of the domain $\Omega = [a, b] \subset \mathbb{R}$
- two approaches to incorporate Dirichlet BCs:
  - **direct**: incorporate when setting up the linear system, i.e.,

$$u^h(x) = \Sigma_{j=2}^{N-1} u_j N_j(x) + u(a)N_1(x) + u(b)N_N(x)$$
$$v^h(x) = \Sigma_{i=2}^{N-1} v_i N_i(x) \,.$$

    $\rightarrow$ yields system matrices of size $N - 2 \times N - 2$
  - **indirect**: ignore when setting up the linear system, i.e.,

$$u^h(x) = \Sigma_{j=1}^{N} u_j N_j(x)$$
$$v^h(x) = \Sigma_{i=1}^{N} v_i N_j(x) \,.$$

    $\rightarrow$ yields system matrices of size $N \times N$
    $\rightarrow$ Dirichlet BCs still need to be incorporated before solving.

Incorporating Dirichlet BCs

There are two main methods to incorporate the Dirichlet BCs before solving the linear system:

- **elimination**: reduce the $N \times N$ system to a $N - 2 \times N - 2$ system for the non-Dirichlet nodes.
- **penalty**: modify the linear system to ensure that the solution of the full $N \times N$ system will satisfy the BCs.

Example: Elimination of one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Incorporate by:

Example: Elimination of one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} g \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Incorporate by:

1. inserting g into linear system

Example: Elimination of one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{12} & K_{13} \\ K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} - \begin{bmatrix} K_{11} \\ K_{21} \\ K_{31} \end{bmatrix} g$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Incorporate by:

1. inserting g into linear system
2. bringing all knowns (i.e., $K_{i1} g$) to the right hand side

Example: Elimination of one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} \cancel{K_{12}} & \cancel{K_{13}} \\ K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cancel{f_1} \\ f_2 \\ f_3 \end{bmatrix} - \begin{bmatrix} \cancel{K_{11}} \\ K_{21} \\ K_{31} \end{bmatrix} g$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Incorporate by:

1. inserting g into linear system
2. bringing all knowns (i.e., $K_{i1}g$) to the right hand side
3. removing the redundant equation (equation for node 1)

Example: Elimination of one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2 - K_{21} g \\ f_3 - K_{21} g \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Incorporate by:

1. inserting g into linear system
2. bringing all knowns (i.e., $K_{i1} g$) to the right hand side
3. removing the redundant equation (equation for node 1)
4. final system has one row and one column less

Example: Penalty approach for one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Let's solve for it anyway by:

Example: Penalty approach for one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} + p & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Let's solve for it anyway by:

1. adding a very large number p to K_{11} and

Example: Penalty approach for one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} + p & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 + pg \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Let's solve for it anyway by:

1. adding a very large number p to K_{11} and
2. adding pg to the corresponding entry of the RHS vector

Example: Penalty approach for one Dirichlet node

- Linear system (without Dirichlet BCs):

$$\begin{bmatrix} K_{11} + p & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 + pg \\ f_2 \\ f_3 \end{bmatrix}$$

- Let $u_1 = g$ be known due to a Dirichlet BC
- Let's solve for it anyway by:

1. adding a very large number p to K_{11} and
2. adding pg to the corresponding entry of the RHS vector
3. solving the modified system

Note: if p is large enough, then

$$\begin{aligned} (K_{11} + p)u_1 + K_{12}u_2 + K_{13}u_3 &\approx pu_1 \,, \\ f_1 + pg &\approx pg \end{aligned} \qquad \Rightarrow \qquad pu_1 \approx pg$$