CAE of Sensors and Actuators
Date: Januar 2021

Lehrstuhl für Sensorik
Universität Erlangen-Nürnberg
Daniel Kiefer

Exercise sheet 3: FE calculus – solving simple linear systems

Solve the boundary value problems (BVPs) given below using FE-calculus.

General Remarks

- We denote with $u(x)$ the dependent variable, while x, y, z denote the independent variables and $u'(x)$ is the derivative with respect to x.
- Assume $u(x)$ and x to be normalized (no unit).
- Solve the 1D-problem on the domain $\Omega = [0, 2]$.
- Use only two elements of length 1.
- Use linear shape functions, i.e.,

$$N_1(x) = \begin{cases} 1 - x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$
$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2 - x & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$
$$N_3(x) = \begin{cases} x - 1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

- For the Galerkin discretization: use $v^h(x)$ as test function and $u^h(x)$ as trial function.

Exercises

Problem 1: solve

$$\text{PDE}: u''(x) + 6u(x) = 2 \quad \text{for} \quad x \in \Omega = [0,2] \tag{1a}$$
$$\text{BCs}: u(0) = 1,\ u(2) = 2 \tag{1b}$$

by incorporating the BCs directly, i.e., use the ansatz

$$u^h(x) = N_1(x)u(0) + N_2(x)u_2 + N_3(x)u(2)$$
$$v^h(x) = N_2(x)v_2\,.$$

Problem 2: solve the BVP (1) but do not consider the BCs directly. Instead, use the ansatz

$$u^h(x) = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3$$
$$v^h(x) = N_1(x)v_1 + N_2(x)v_2 + N_3(x)v_3\,.$$

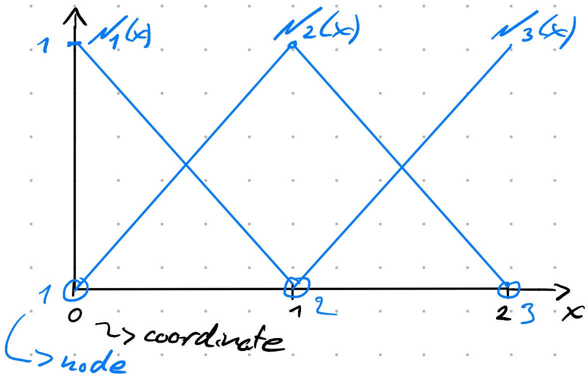
Subsequently, apply the elimination approach to impose the Dirichlet BCs.

Problem 3: solve again the same BVP (1). This time use linear reference elements instead of the global hat functions $N_i(x)$. Compute first the element matrices/vectors, then assemble them into the global ones. Incorporate the Dirichlet BCs by applying the elimination approach and solve the reduced system.

Exercise sheet 3

PDE: $u'' + 6u = 2$ on $x \in \Omega = [0,2]$

BC: $u(0) = 1$, $u(2) = 2$



$$N_1(x) = \begin{cases} 1-x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2-x & x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

Problem 1

- include BCs in FE-ansatz and solve:

1. Derive the weak form:

$$\int_0^2 v u'' + 6vu \, dx = \int_0^2 v \, 2 \, dx$$

$$\underbrace{[\cancel{v u'}]_0^2}_{0 \to DBC} - \int_0^2 v'u' \, dx + \int_0^2 6 v u \, dx = \int_0^2 2 v \, dx$$

2. Ansatz for Galerkin discretization:

$$v \approx v^h(x) = \sum_i v_i N_i(x) = v_2 N_2(x)$$

$$u \approx u^h(x) = \sum_j u_j N_j(x) = 1 N_1(x) + u_2 N_2(x) + 2 N_3(x)$$

3. insert into weak form:

$$\underbrace{-\int_0^2 \cancel{v_2} N_2' (N_1' + u_2 N_2' + 2N_3') \, dx}_{A} + \underbrace{\int_0^2 6 \cancel{v_2} N_2 (N_1 + u_2 N_2 + 2N_3) \, dx}_{B}$$

$$= \underbrace{\int_0^2 2 \cancel{v_2} N_2 \, dx}_{F}$$

$$A = -\int_0^1 1(-1 + u_2 1 + 2\cdot 0) \, dx - \int_1^2 (-1)(0 + u_2(-1) + 2(+1)) \, dx$$

$$= -\int_0^1 u_2 - 1 \, dx - \int_1^2 u_2 - 2 \, dx$$

$$= -[(u_2 - 1)x]_0^1 - [(u_2-2)x]_1^2$$

$$= -(u_2 - 1) - (u_2 - 2) = -2u_2 + 3$$

$$B = \underbrace{\int_0^2 6 \cancel{v_2} N_2 (N_1 + u_2 N_2 + 2N_3)\, dx}_{B}$$

$$N_1(x) = \begin{cases} 1-x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2-x & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$B = \int_0^1 6x(1-x + u_2(x) + 2 \cdot 0\, dx + \int_1^2 6(2-x)(0 + u_2(2-x) + 2(x-1)\, dx$$

$$\vdots$$

$$= 3 + 4u_2$$

$$F = \underbrace{\int_0^2 2 \cancel{v_2} N_2\, dx}_{F}$$

$$F = \int_0^1 2x\, dx + \int_1^2 2(2-x)\, dx$$

$$= 2$$

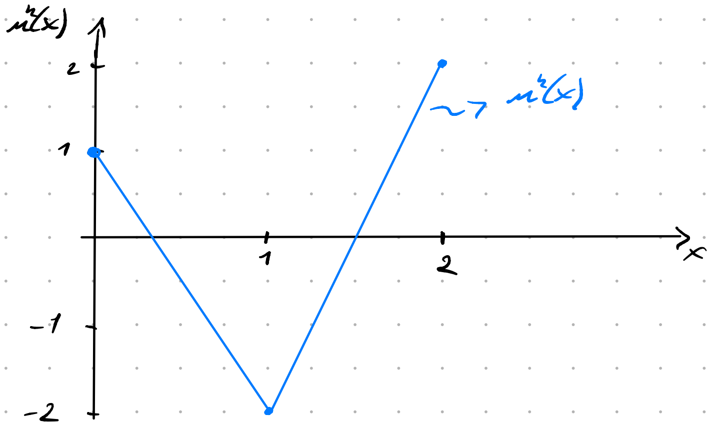
- plug back into equation: $A + B = F$

$$-2u_2 + 3 + 3 + 4u_2 = 2$$

$$\Rightarrow \quad 2u_2 = -4$$

$$\Rightarrow \quad u_2 = -2 \quad // \quad , \text{ DBC values: } u_1 = 1, \quad u_3 = 2$$

- plot solution:



Problem 2 (Note: you can ignore the blue parts)

- first derive Galerkin discretization, then incorporate BCs.

ansatz: $v^h = \sum_{i=1}^{3} v_i N_i(x)$, $u^h = \sum_{j=1}^{3} u_j N_j(x)$

weak form: $-\int_0^2 v' u' \, dx + \int_0^2 6 v u \, dx = \int_0^2 2 v \, dx - [v u']_0^2$

expressions at boundary: $u^h(2) = \sum_j u_j \delta_{j3}$, $u^h(0) = \sum_j u_j \delta_{j1}$, $v^h(2) = \sum_i v_i \delta_{i3}$, $v^h(0) = \sum_i v_i \delta_{i1}$

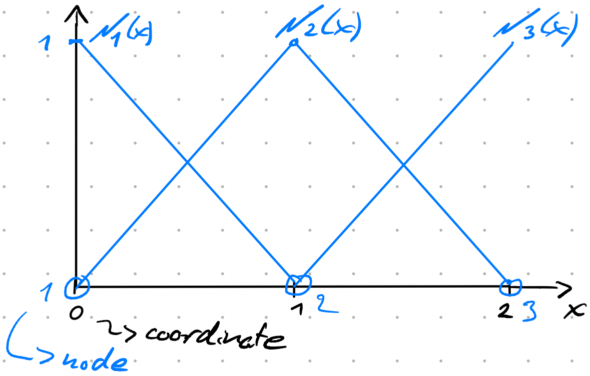
$\Rightarrow u^{h\prime}(0) = \frac{u_2 - u_1}{1} = \sum_j u_j(\delta_{j2} - \delta_{j1})$, $u^{h\prime}(2) = \frac{u_3 - u_2}{1} = \sum_j u_j(\delta_{j3} - \delta_{j2})$ $\rightsquigarrow$ slope with $x_{i+1} - x_i = 1$

insert ansatz into weak form

$$\sum_i v_i \sum_j \left[\int_0^2 - N_i' N_j' \, dx + \int_0^2 6 N_i N_j \, dx \right] u_j + \sum_i v_i \sum_j \left[\delta_{i3}(\delta_{j3} - \delta_{j2}) - \delta_{i1}(\delta_{j2} - \delta_{j1}) \right] u_j = \sum_i v_i \int_0^2 2 N_i \, dx$$

test functions $v^h(x)$ are arbitrary $\rightsquigarrow$ eliminate v_i and $\sum_i$:

$$\sum_j \Big[\underbrace{\int_0^2 - N_i' N_j' \, dx}_{A_{ij}} + \underbrace{\int_0^2 6 N_i N_j \, dx}_{B_{ij}} + \underbrace{\delta_{i3}(\delta_{j3} - \delta_{j2}) - \delta_{i1}(\delta_{j2} - \delta_{j1})}_{C_{ij}} \Big] u_j = \underbrace{\int_0^2 2 N_i \, dx}_{F_i}$$



$$N_1(x) = \begin{cases} 1 - x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & x \in [0,1] \\ 2 - x & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x - 1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$\left.\begin{aligned} A_{11} &= \int_0^1 -(-1)(-1) \, dx + \int_1^2 0 \, dx = -1 \\ A_{22} &= \int_0^1 -(1)(1) \, dx + \int_1^2 -(-1)(-1) \, dx = -2 \\ A_{12} &= \int_0^1 -(-1)(1) \, dx + \int_1^2 -(0)(-1) \, dx = 1 \end{aligned}\right\} \begin{array}{l} \text{sym!} \\ \text{sparse!} \end{array} \rightsquigarrow A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B_{11} = 6\int_0^1 (1-x)(1-x) \, dx = 6 \int_0^1 1 - 2x + x^2 \, dx = 6 \left[x - 2x^2/2 + x^3/3 \right]_0^1 = 2$$

$$B_{12} = 6 \int_0^1 (1-x)(x) \, dx + 6 \int_1^2 (0)(2-x) \, dx = 6 \int_0^1 (x - x^2) \, dx = 6 \left[x^2/2 - x^3/3 \right]_0^1 = 1$$

$$B_{22} = 6 \int_0^1 (x^2) \, dx + 6 \int_1^2 (2-x)(2-x) \, dx = 6 \int_0^1 x^2 \, dx + 6 \int_1^2 (4 - 4x + x^2) \, dx$$
$$= 6 [x^3/3]_0^1 + 6 \left[4x - 2x^2 + x^3/3 \right]_1^2 = 2 + 16 - 12 - 2 = 4$$

$$\rightsquigarrow \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$\rightsquigarrow$ because

δ_{i3} = 3rd row is 1, other 0.

δ_{j1} = 1st column is 1, other 0.

etc...

$$F_1 = \int_0^1 2(1-x)\,dx + \int_1^2 2(0)\,dx = 2\left[x - x^2/2\right]_0^1 = 1$$
$$F_2 = \int_0^1 2(x)\,dx + \int_1^2 2(2-x)\,dx = 2\left[x^2/2\right]_0^1 + \left[4x - x^2\right]_1^2 = 1 + 4 - 3 = 2$$
$$F_3 = F_1 = 1 \quad , \quad F = [1, 2, 1]^T$$

computed matrices:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

equation: $(A + B + C) \cdot u = F$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad \text{linear system}$$

- include BCs:

$u_1 = 1 , \quad u_3 = 2$

use elimination approach:

$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} 2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{1} \\ 2 \\ \cancel{1} \end{pmatrix} u_2 = \begin{pmatrix} \cancel{1} \\ 2 \\ \cancel{1} \end{pmatrix} - \begin{pmatrix} \cancel{2} \\ 2 \\ \cancel{0} \end{pmatrix} - \begin{pmatrix} \cancel{0} \\ 4 \\ \cancel{4} \end{pmatrix} = \begin{pmatrix} \cancel{-1} \\ -4 \\ \cancel{-3} \end{pmatrix}$$

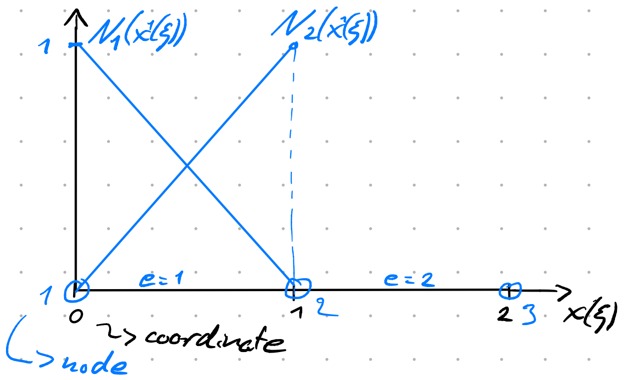
$$\rightsquigarrow \quad 2u_2 = -4$$
$$\Rightarrow \quad u_2 = -2 \quad //$$

Note: the boundary term does not need to be set up because only eq. 2 is of interest. The boundary equations need to be eliminated in the elimination approach.

Problem 3

- use linear reference elements

$$\sum_j \Big[\underbrace{\int_0^2 - N_i' N_j' \, dx}_{A_{ij}} + \underbrace{\int_0^2 6 N_i N_j \, dx}_{B_{ij}} \Big] u_j = \underbrace{\int_0^2 2 N_i \, dx}_{F_i} \quad \forall i$$



Element shape functions:

$$\begin{cases} N_1(\xi) = 1-\xi \\ N_2(\xi) = \xi \end{cases}, \quad \text{where } \xi \in [0,1]$$

$A_{ij} = \sum_e A_{ij}^e$, $i, j \in [1 \dots \text{number of nodes}]$

-> reduced element matrices / vectors: A_{ab}^e , $a, b \in [1 \dots$ number of element nodes$]$
-> assemble A_{ab}^e into A_{ij}

Compute red. elem. matrices: $J_e = \frac{\partial x(\xi)}{\partial \xi} = h_e = 1$

$$A_{ab}^e = \int_0^1 (-1) N_a' N_b' \, \cancel{\det J_e}^{1} \, d\xi \overset{a=b}{=} \int_0^1 (-1)\, 1 \, d\xi = [_0^1 \xi] = -1$$

$$\overset{a \neq b}{=} \int_0^1 (-1)(1)(-1) \, d\xi = +1$$

$$[A^e] = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$[A] = \begin{bmatrix} -1 & +1 & 0 \\ +1 & -2 & +1 \\ 0 & +1 & -1 \end{bmatrix}$$

$$B_{ab}^e = \int_0^1 6 N_a N_b \, \cancel{\det J_e}^{1} \, d\xi \overset{a=b}{=} \int_0^1 6 \xi^2 \, d\xi = \frac{6}{3} [\xi^3]_0^1 = 2$$

$$\overset{a \neq b}{=} \int_0^1 6 \xi (1-\xi) \, d\xi = [3\xi^2 - 2\xi^3]_0^1 \; 1$$

$$[B^e] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$F_a^e = \int_0^1 2 N_a \, \cancel{\det J_e}^{1} \, d\xi \overset{a=1}{=} \int_0^1 2(1-\xi) \, d\xi = [2\xi - \xi^2]_0^1 = 1$$

$$\overset{b=2}{=} \int_0^1 2\xi \, d\xi = [\xi^2]_0^1 = 1$$

$$[F^e] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[F] = [1, \; 2, \; 1]^T$$

- total eq. : $(\underline{\underline{A}} + \underline{\underline{B}}) \cdot \underline{u} = \underline{F}$

- We need only eq. for node # 2 (node #1 and #3 are DBCs):

$$\left[(1 \quad -2 \quad 1) + (1 \quad 4 \quad 1) \right] \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2$$

with $u_1 = 1$, $u_3 = 2$

$$\left[2 \quad 2 \quad 2 \right] \cdot \begin{pmatrix} 1 \\ u_2 \\ 2 \end{pmatrix} = 2$$

$\Rightarrow$ solve for u_2:

$$2 + 2u_2 + 4 = 2$$

$$\Rightarrow u_2 = -2$$