Lehrstuhl für Sensorik
Universität Erlangen-Nürnberg
Daniel Kiefer

CAE of Sensors and Actuators
Date: January 2021

Exercise sheet 2: Derivation of linear systems

General Remarks

- We denote with $u(x)$ the dependent variable, while x, y, z denote the independent variables. $u'(x)$ is the derivative with respect to x.
- Assume $u(x)$ and x to be normalized (no unit).
- Use N nodes for the derivations (N-1 elements).
- Denote the shape functions with N_i and N_j.
- v_i and u_j shall represent the nodal values $v^h(x_i)$ and $u^h(x_j)$, respectively.

Exercises

Problem 1: Let the weak formulation be given as

$$\text{weak}: \int_a^b v'u' + cvu' \, \mathrm{d}x = \int_a^b vf \, \mathrm{d}x \quad \text{on} \quad x \in \Omega = [a, b] \tag{1a}$$

$$\text{DBCs}: u(a) = g_a, \; u(b) = g_b \, . \tag{1b}$$

Derive the discrete Galerkin formulation by using the following Finite Element approximation:

$$v(x) \approx v^h(x) = \sum_{i=2}^{N-1} N_i(x) v_i \, , \quad u(x) \approx u^h(x) = \sum_{j=2}^{N-1} N_j(x) u_j + N_1(x) g_a + N_N(x) g_b \, .$$

This is done by putting v^h and u^h into the weak form (1) and resorting the terms to obtain a system of $N-2$ linear equations of the form

$$\sum_{j=2}^{N-1} S_{ij} u_j = f_i - S_{i1} g_a - S_{iN} g_b \quad \forall i \in [2, N-1] \subset \mathbb{Z} \, .$$

Write down the resulting expressions for the system matrix S_{ij}, the excitation vector f_i and the Dirichlet contributions S_{i1} and S_{iN}.

Problem 2: Let the weak formulation be given as

$$\text{weak}: \int_a^b v'u' \, \mathrm{d}x = \int_a^b vf \, \mathrm{d}x + v(b) h_b \quad \text{on} \quad x \in \Omega = [a, b] \tag{2a}$$

$$\text{DBC}: u(a) = g_a \, . \tag{2b}$$

Derive the discrete Galerkin formulation by using the following Finite Element approximation:

$$v(x) \approx v^h(x) = \sum_{i=2}^{N} N_i(x) v_i \, , \quad u(x) \approx u^h(x) = \sum_{j=2}^{N} N_j(x) u_j + N_1(x) g_a \, .$$

This is done by putting v^h and u^h into the weak form (2) and resorting the terms to obtain a system of $N-1$ linear equations of the form

$$\sum_{j=2}^{N} S_{ij} u_j = f_i - S_{i1} g_a \quad \forall i \in [2, N] \subset \mathbb{Z} \, .$$

Write down the resulting expressions for the system matrix S_{ij}, the excitation vector f_i and the Dirichlet contribution S_{i1}.

Exercise sheet 2

Problem 1

weak form: $\int_\Omega v'u' + c v u' \, dx = \int_\Omega v f \, dx$ ①

DBC: $u(a) = g_a$, $u(b) = g_b$

ansatz: $v^h(x) = \sum_{i=2}^{N-1} N_i(x) v_i$

$u^h(x) = \sum_{j=2}^{N-1} N_j(x) u_j + N_1(x) g_a + N_N(x) g_b$

1. put v^h and u^h into ①:

$$\int_\Omega \Big(\sum_{i=2}^{N-1} N_i' v_i\Big)\Big(\sum_{j=2}^{N-1} N_j' u_j + N_1' g_a + N_N' g_b\Big) dx$$
$$+ \int_\Omega c \Big(\sum_{i=2}^{N-1} N_i v_i\Big)\Big(\sum_{j=2}^{N-1} N_j' u_j + N_1' g_a + N_N' g_b\Big) dx$$
$$= \int_\Omega \Big(\sum_{i=2}^{N-1} N_i v_i\Big) f \, dx$$

2. Move sums and u_j, v_i out of the integrals:

$$\sum_{i=2}^{N-1} v_i \Big[\sum_{j=2}^{N-1} \int_\Omega N_i' N_j' \, dx \, u_j + \int_\Omega N_i' N_1' \, dx \, g_a + \int_\Omega N_i' N_N' \, dx \, g_b \Big]$$
$$+ \sum_{i=2}^{N-1} v_i \Big[\sum_{j=2}^{N-1} \int_\Omega c N_i N_j' \, dx \, u_j + \int_\Omega c N_i N_1' \, dx \, g_a + \int_\Omega c N_i N_N' \, g_b \Big]$$
$$= \sum_{i=2}^{N-1} v_i \Big[\int_\Omega N_i f \, dx \Big]$$

3. v_i are arbitrary

$$\sum_{j=2}^{N-1} \overbrace{\int_\Omega N_i' N_j' \, dx}^{(N-2)\times(N-2) \text{ coefficients}} u_j + \int_\Omega N_i' N_1' \, dx \, g_a + \int_\Omega N_i' N_N' \, dx \, g_b$$
$$+ \sum_{j=2}^{N-1} \int_\Omega c N_i N_j' \, dx \, u_j + \int_\Omega c N_i N_1' \, dx \, g_a + \int_\Omega c N_i N_N' \, dx \, g_b$$
$$= \int_\Omega N_i f \, dx$$

4. reorder terms: move all known quatities to the right

$$\sum_{j=2}^{N-1} \underbrace{\int_\Omega N_i' N_j' + c N_i N_j \, dx}_{S_{ij}} \, u_j = \underbrace{\int_\Omega N_i f \, dx}_{f_i} +$$

$$- \underbrace{\int_\Omega N_i' N_1' + c N_i N_1 \, dx}_{S_{i1}} \, g_a - \underbrace{\int_\Omega N_i' N_N' + c N_i N_N \, dx}_{S_{iN}} \, g_b$$

5. symbolic notation: $\underline{\underline{S}} \cdot \underline{u} = \underline{f} - \underline{S}_1 g_a - \underline{S}_N g_b$

Problem 2

① weak: $\int_\Omega v' u' \, dx = \int_\Omega v f \, dx + v(b) h_b$ on $x \in \Omega = (a,b)$

$\rightsquigarrow v^h(b) = \sum_{i=2}^{N} \delta_{iN} v_i = v_N$

with $\delta_{iN} = \begin{cases} 1 & i = N \\ 0 & i \neq N \end{cases}$

DBC: $u(a) = g_a$

FE ansatz: $v^h(x) = \sum_{i=2}^{N} N_i(x) v_i + \cancel{v_1}^{0} N_1(x)$

$u^h(x) = \sum_{j=2}^{N} N_j(x) u_j + N_1 g_a$

1. plug into ①:

$$\int_a^b \sum_{i=2}^{N} N_i' v_i \left(\sum_{j=2}^{N} N_j' u_j + N_1' g_a \right) dx = \int_a \left(\sum_{i=2}^{N} N_i v_i \right) f \, dx + \sum_{i=2}^{N} \delta_{iN} v_i h_b$$

2. move sums and u_j, v_i out of the integrals:

$$\sum_{i=2}^{N} v_i \left[\sum_{j=2}^{N} \int_a^b N_i' N_j' \, dx \, u_j + \int_a^b N_i' N_1' \, dx \, g_a \right] = \sum_{i=2}^{N} v_i \left[\int_a^b N_i f \, dx + \delta_{iN} h_b \right]$$

3. v_i can be arbitrary $\rightsquigarrow$ remove from eq.:

$$\sum_{j=2}^{N} \int_a^b N_i' N_j' \, dx \, u_j + \int_a^b N_i' N_1' \, dx \, g_a = \int_a^b N_i f \, dx + \delta_{iN} h_b$$

4. move known terms to the right:

node 'N' is at the Neumann boundary

$$\sum_{j=2}^{N} \underbrace{\int_a^b N_i' N_j' \, dx}_{S_{ij}} u_j = \underbrace{\int_a^b N_i f \, dx}_{f_i} + \underbrace{\delta_{iN} h_b}_{N_i} - \underbrace{\int_a^b N_i' N_1' \, dx}_{S_{i1}} g_a$$

N_i ↓ Neumann contribution

S_{i1} ↓ Dirichlet contribution