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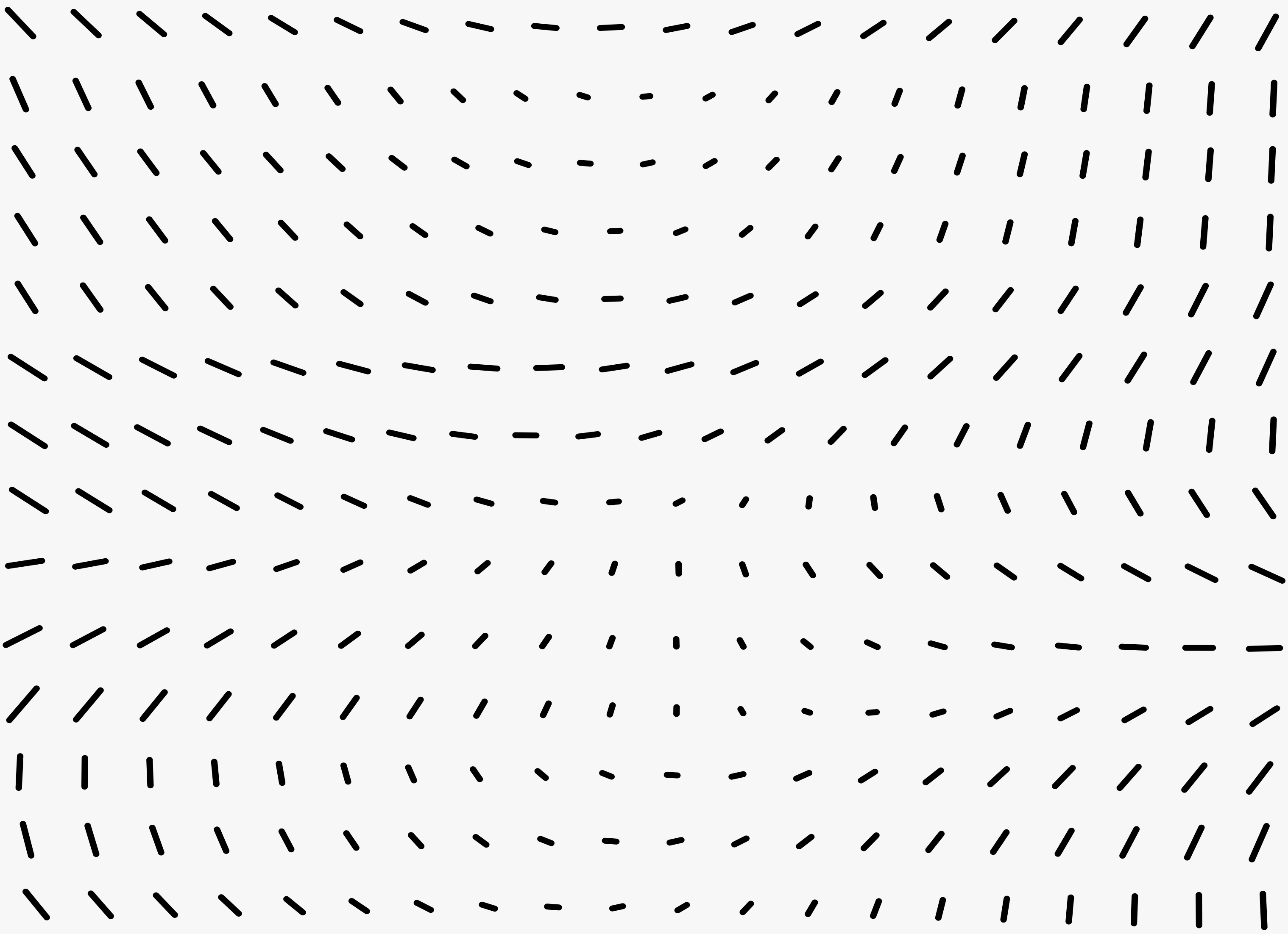
# CAE of Sensors and Actuators

# — Exercise class



# CAE of Sensors and Actuators

## Introduction

CAE: Computer Aided Engineering

Finite Element Method (FEM) → used to solve "field problems"
→ this course is all about FEM

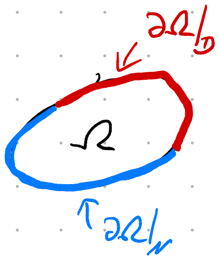
## Field theory

Field:
- physical quantity $u(x, t)$ — $x$: space, $t$: time
- scalar, vector, tensor
- transcient $u(x,t)$, static $u(x)$, harmonic $u(x,\omega)$ ($\omega$: frequency), modal $u(x,\omega)$

- Described by partial differential equations (PDEs)
  - $u$: "dependent variable"
  - $x, t$: "independent variables"

\+ boundary conditions
- → Dirichlet BC (DBC): $u$ is given
- → Neumann BC (NBC): derivative $u'$ is prescribed

} Boundary value problem (BVP)




- Note: DBC and NBC are not independent!
  Example: reformulate fixation of beam (DBC) as reaction forces (NBC):

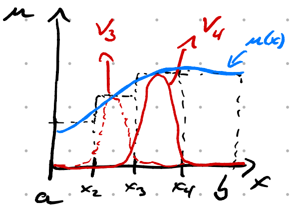

## FE-Method: overview / scheme

$$\boxed{\text{BVP}} \xrightarrow{\text{variation}} \boxed{\text{weak form}} \xrightarrow{\text{discretization}} \boxed{\text{algebraic system}} \xrightarrow{\text{numerical methods}} \boxed{\text{solution}}$$

## Weak form of BVP:

(1) $\begin{cases} \text{PDE:} & k\,u''(x) + f = 0 \quad \text{on } \Omega = [a,\ b] \\ \text{BCs:} & u(a) = g_a \ , \quad u(b) = g_b \ \ (\text{or } u'(b) = h_b \text{ instead}) \end{cases}$

**Basic idea:** require equations to hold in "average" on given intervals:




- integrate (average) over each section:

$a$-$x_2$: $\int_a^{x_2} k\,u''(x) + f \, dx = 0$

$x_2$-$x_3$: $\int_{x_2}^{x_3} k\,u''(x) + f \, dx = 0$

$x_3$-$x_4$: etc... for all intervalls

-> system of equations!
-> as many equations as intervals
-> $\infty$ many $\hat{=}$ PDE

improved idea: instead of intervals, use a weighting function $v(x)$

$$\Rightarrow \quad \int_a^b v(x)\left(k\,u''(x) + f\right) dx = 0$$

↳ The same as before, if we choose unit-height rectangular functions for each interval as $v(x)$.

↳ but more general as before! -> We can choose almost "arbitrary" functions $v_i(x)$

## Derive a weak form of a PDE: (example for eq. (1))

1. multiply with arbitrary <u>test function</u> $v(x)$ which
   - is zero on the Dirichlet boundaries
   - is sufficiently smooth

$$\Rightarrow \quad v(x)\left(k\,u''(x) + f\right) = 0 \quad \text{on } \Omega$$

2. integrate over the whole domain:

$$\Rightarrow \quad \int_a^b v(x)\left(k\,u''(x) + f\right) dx = 0 \qquad \forall\, v(x) \in V$$

↳ are zero at DBC

3. integration by parts (apply integral theorem):

$$\Rightarrow \quad -\int_a^b k\,v'(x)\,u'(x)\,dx + k\left[v\,u'\right]_a^b + \int_a^b v(x)\,f\,dx = 0$$

$$\Rightarrow \quad \int_a^b k\,v'(x)\,u'(x)\,dx = \int_a^b v(x)\,f\,dx + k\overset{0\ (DBC)}{\cancel{v(b)}}\,u'(b) - k\overset{0\ (DBC)}{\cancel{v(a)}}\,u'(a)$$

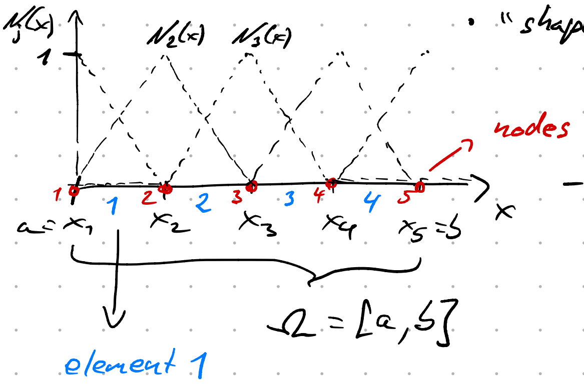
4. include BCs:

$$\Rightarrow \quad \int_a^b k\,v'(x)\,u'(x)\,dx = \int_a^b v(x)\,f\,dx + \overset{\to NBC}{k\,v(b)\,h_b}$$

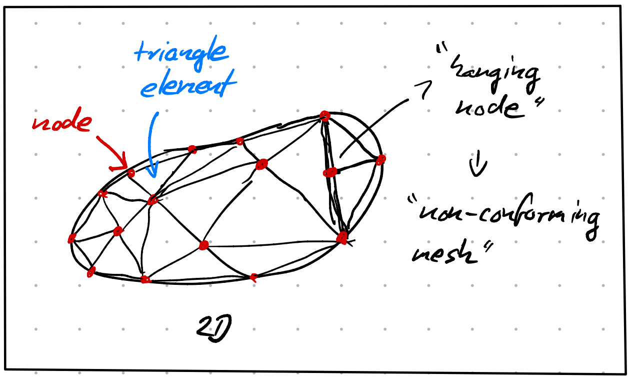
DONE ☺

# Discretization

hat-functions: → in general:
- "ansatz functions"
- "shape functions"




→ FE mesh in 1D




## Galerkin discretization:

$V^h := \{N_i(x)\}$, $U^h := \{N_j(x)\}$ ⇝ finite set of functions used to approximate $u(x)$ and $v(x)$.

- ansatz for $u(x) \approx u^h(x) = \sum_j u_j N_j(x)$

  $v(x) \approx v^h(x) = \sum_i v_i N_i(x)$

  ↓ approximation

> Note: "i" and "j" are "dummy" indexes, i.e., placeholders. We could rename them at any time.

- insert $u^h(x)$ and $v^h(x)$ into the weak form:

$$+\int_a^b k\, v^{h\prime}(x)\, u^{h\prime}(x)\, dx = \int_a^b v^h(x)\, f\, dx \qquad (1)$$

$$\Rightarrow \quad +\int_a^b k \Big(\sum_i v_i N_i'(x)\Big)\Big(\sum_j u_j N_j'(x)\Big) dx = \int_a^b \Big(\sum_i v_i N_i(x)\Big) f\, dx$$

$$\Rightarrow \quad \sum_i v_i \sum_j \underbrace{\Big(+\int_a^b k N_i'(x) N_j'(x)\, dx\Big)}_{K_{ij}} u_j = \sum_i v_i \underbrace{\int_a^b f N_i(x)\, dx}_{f_i}$$

$$\Rightarrow \quad \sum_i v_i \Big(\sum_j K_{ij}\, u_j\Big) = \sum_i v_i f_i \quad \text{, with } v_i \text{ arbitrary!}$$

$$\Rightarrow \quad \sum_j K_{ij}\, u_j = f_i$$

$$\Rightarrow \quad \underline{\underline{K}} \cdot \underline{u} = \underline{f} \qquad \text{(system of linear equations)} \qquad (2)$$

⇝ discrete approx. to (1).

$\underline{\underline{K}}$ : "stiffness matrix"

$\underline{u}$ : "vector of unknows", "nodal degrees of freedom (DoF)"

$\underline{f}$ : "force vector"

⇝ terminology from mechanics

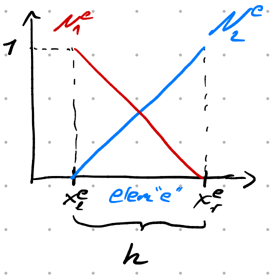
⇝ Once we computed the entries $K_{ij}$ and $f_i$, we can solve (2) for the unknowns $u_j$.

# Per element computation of $K_{ij}$ and $f_i$

half the hat-functions appear on the element domain $\Omega_e = [x_\ell^e, x_r^e]$

$$N_1^e(x) = \frac{(x_r^e - x)}{x_r^e - x_\ell^e} = \frac{1}{h}(x_r^e - x)$$

$$N_2^e(x) = \frac{(x - x_\ell^e)}{x_r^e - x_\ell^e} = \frac{1}{h}(x - x_\ell^e)$$




element size: $h = x_r^e - x_\ell^e$

"assembling" (will be done below)

"element stiffness"

is zero for all $i,j$ except 4 of them ↳ reduce to $k_{ab}^e$

Global stiffness: $$K_{ij} = \sum_e \int_{\Omega_e} k N_i'(x) N_j' dx, \quad i,j \in \{1 \dots N_n\} \text{ (global node numbers)}$$

**Element stiffness:** $$k_{ab}^e = \int_{\Omega_e} k N_a'(x) N_b'(x) dx, \quad a,b \in \{1,2\} \text{ (elem. nodes)}$$

$$\left.\begin{aligned} k_{21}^e &= k \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx = -\frac{k}{h} \\ k_{12}^e &= k_{21}^e \\ k_{11}^e &= k \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx = +\frac{k}{h} \\ k_{22}^e &= k_{11}^e \end{aligned}\right\} \quad \underline{\underline{k}}^e = \frac{k}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global force: $$F_i = \sum_e \int_{\Omega_e} f N_i(x) dx, \quad i \in \{1 \dots N_n\}$$

**Element force:** $$f_a^e = \int_{\Omega_e} f N_a(x) dx, \quad a \in \{1,2\}$$

$$\left.\begin{aligned} f_1^e &= \int_{x_\ell^e}^{x_r^e} f N_1^e(x) dx = \int_{x_\ell^e}^{x_r^e} f \frac{(x_r^e - x)}{h} dx = \frac{h}{2} \\ f_2^e &= \int_{x_\ell^e}^{x_r^e} f N_2^e(x) dx = \int_{x_\ell^e}^{x_r^e} f \frac{(x - x_\ell^e)}{h} dx = \frac{h}{2} \end{aligned}\right\} \quad \underline{f}^e = \frac{h}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**Assemble $\underline{\underline{k}}^e$ into $\underline{\underline{K}}$:**

for each element "e":

1. compute $k_{ab}^e$
2. map $(a,b) \mapsto (i,j)$ (local to global node indices)
3. set $K_{ij} \leftarrow K_{ij} + k_{ab}^e$

Assemble $\underline{f}^e$ into $\underline{F}$ in the same way (you need only one index).

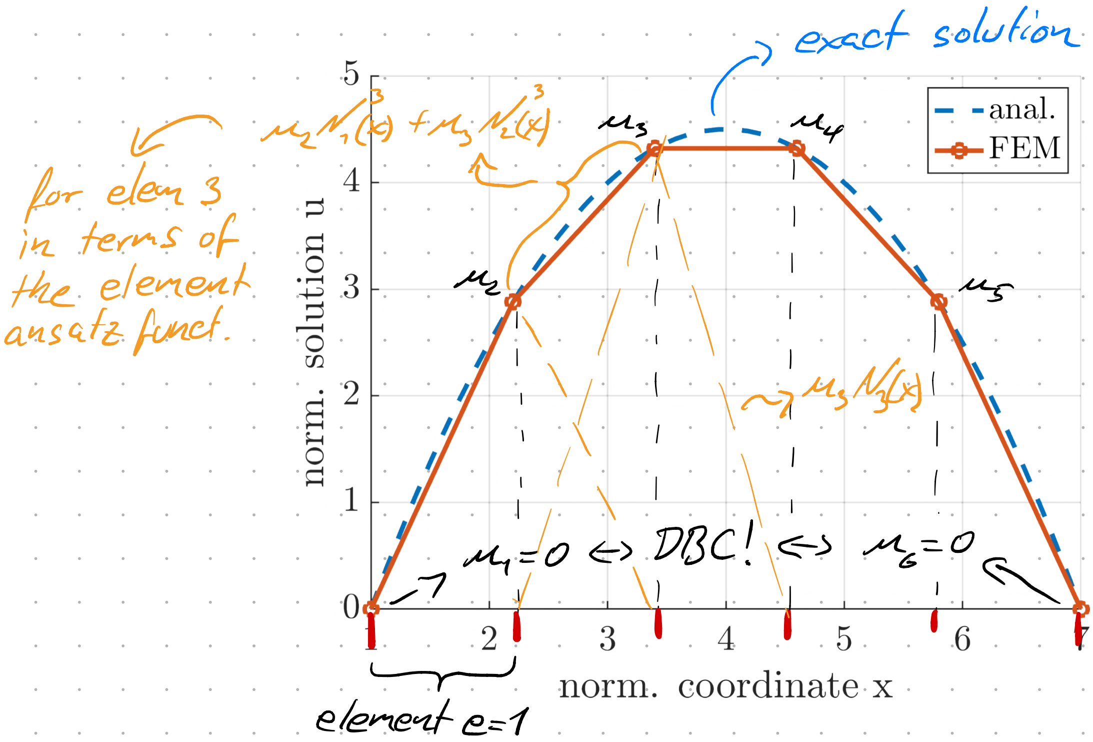
**Solve $\underline{\underline{K}} \cdot \underline{u} = \underline{F}$** on all <u>except the Dirichlet nodes</u>! ⤳ $u$ is known there.

↳ realizes a homogeneous DBC, i.e., $u(x)|_{\partial\Omega_D} = 0$.

# Example solution for Assignment A0

$$\begin{cases} -1\,u''(x) = 1 & \text{on } x \in \Omega = [1, 7] \\ u(a) = 0, \quad u(b) = 0 \end{cases}$$

using $Ne = 5$ (number of elements):




in total $u_{FEM}(x) = \sum_{j=2}^{5} u_j N_j(x)$ $\quad (u_1 = 0,\ u_6 = 0)$

# Electrostatics

from Maxwell equations:

Faraday induction: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Gauss law: $\nabla \cdot \vec{D} = q_e$ $\rightsquigarrow$ $\vec{D}$ begins and ends at charges!

1. assume $\frac{\partial \vec{B}}{\partial t} \overset{!}{=} 0$

$$\Rightarrow \begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{D} = q_e \end{cases}$$

2. material law: $\vec{D} = [\varepsilon] \vec{E}$

$$\nabla \times \vec{E} = 0$$
$$\nabla \cdot ([\varepsilon]\vec{E}) = q_e$$

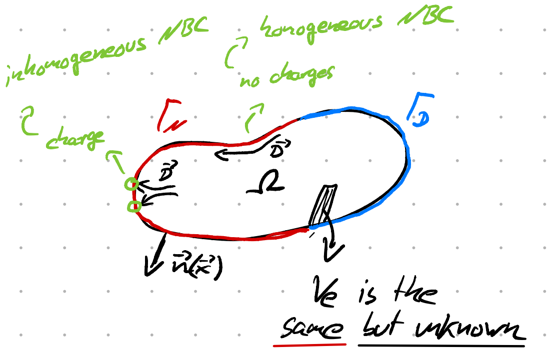
$\rightsquigarrow$ How to solve both equations simultaneously?

3. Satisfy $\nabla \times \vec{E}$ implicitly by introducing the electrostatic potential $V_e$: $\vec{E} \overset{\text{def}}{=} -\nabla V_e(\vec{x}) = -\begin{pmatrix} \frac{\partial V_e}{\partial x} \\ \frac{\partial V_e}{\partial y} \\ \frac{\partial V_e}{\partial z} \end{pmatrix}$

$$\Rightarrow -\nabla \cdot ([\varepsilon] \nabla V_e) = q_e$$

↳ Electrostatic PDE

## Electrostatic boundary conditions




- Dirichlet BC: $V_e = a$ on $\Gamma_D$
- Neumann BC: $\nabla V_e \cdot \vec{n} = b$ on $\Gamma_N$

  equivalently: $-[\varepsilon] \nabla V_e \cdot \vec{n} = c$ on $\Gamma_N$

  $\vec{D} \cdot \vec{n} = c$ on $\Gamma_N$
- constraint (floating potential)

  ↳ can be on domain or boundary level

## When to use electrostatics

charge relaxation time $\tau = \frac{\varepsilon}{\sigma}$ $\rightsquigarrow$ electrical permittivity, $\sigma$ $\rightsquigarrow$ electrical conductance $\Big\}$ Cu $\rightsquigarrow$ $\tau = 10^{-15}$ s; silica glas $\rightsquigarrow$ $\tau = 10^{3}$ s

$\rightsquigarrow$ use electrostatics for insulators! so that $t \ll \tau$

Verify $\nabla \times \vec{E} = 0$ for $\vec{E} = -\nabla V_e$

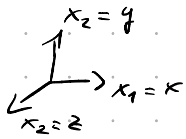
$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} \partial V_e/\partial x \\ \partial V_e/\partial y \\ \partial V_e/\partial z \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 V_e}{\partial y \partial z} - \frac{\partial^2 V_e}{\partial z \partial y} \\ \frac{\partial^2 V_e}{\partial z \partial x} - \frac{\partial^2 V_e}{\partial x \partial z} \\ \frac{\partial^2 V_e}{\partial x \partial y} - \frac{\partial^2 V_e}{\partial y \partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \forall\, V_e(\vec{x}, t)$$

# Mechanical Field

gradient of vector is taken component-wise and yields 2nd order tensor.

- displacements $\vec{u}$ : 1st order tensor $\simeq$ vector
- strain: $s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \rightsquigarrow [s] = \frac{1}{2}\left(\overbrace{\nabla\vec{u}} + (\nabla\vec{u})^T\right)$

  strain tensor $[s] = [s_{ij}] = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix}$




  -> 2nd order tensor
  -> symmetric

- stress tensor: $[\sigma] = [\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$

  -> 2nd order tensor
  -> describes the local inner forces in the body
  -> is symmetric for balance of angular momentum
  -> tractions: $\vec{t} = [\sigma]^T \cdot \vec{n}$

  $\vec{b} = \underset{\text{matrix}}{\overleftarrow{A}} \cdot \vec{a}$

- strain-stress relation / material law
  -> linear: Hook's law: $[\sigma] = [c]:[s]$

  "double contraction" $\simeq$ multiplication of 4th with 2nd order tensors.

  ↳ stiffness tensor (4th order)
  $[c] = [c_{ijkl}]$

  Note: due to symmetries of $[c]$:
  $[\sigma] = [c]:[s] = [c]:\underbrace{\nabla\vec{u}}_{\text{grad } \vec{u}}$ !

## Voigt notaded mechanics

- avoid tensors
- Voigt notation: exploits the symmetry of strain and stress
  - $[s]$ -> 6×1-vector $s$
  - $[\sigma]$ -> 6×1-vector $\sigma$ ← (no brackets!)
  - $[c]$ -> 6×6-matrix $c$

  -> equations of motion require the definition of a new matrix differentiation operator: "$\beta$-operator" (size 6×3)
  ↳ such that $s = \beta \vec{u}$

- Equations of motion in Voigt-notation:

$$\underbrace{\beta^T \cdot \sigma}_{\text{divergence of the stresses}} + \underbrace{\vec{f_v}}_{\text{local "forces" (exterior)}} = \underbrace{\rho \ddot{\vec{u}}}_{\text{inertial "forces"}}$$

- Navier's equation (dependent variable is $\vec{u}$):

$$\beta^T \cdot c \cdot \beta \vec{u} + \vec{f_v} = \rho \ddot{\vec{u}}$$

$\rightsquigarrow$ PDE solved in Comsol (transient study)

- free harmonic motion (modal study)

→ ansatz: $\vec{u} = \vec{\Phi}(\vec{x})\, e^{j\omega t}$ ⇝ $\ddot{\vec{u}} = -\omega^2 \vec{\Phi} e^{j\omega t}$

→ free motions $\vec{f}_V = \vec{0}$ and only homogeneous BCs are allowed

⇒ $\mathcal{B}^T \cdot c \cdot \mathcal{B} \cdot \vec{\Phi} = -\omega^2 \varrho \vec{\Phi}$

⇝ for which values of $(\omega^2, \vec{\Phi})$ is above equation fulfilled?

↳ eigenvalue problem for eigenvalue $\lambda = -\omega^2 = (j\omega)^2$

↳ solutions $\omega_i$ : angular frequencies

$\vec{\Phi}$ : eigenvectors (mode shapes)

## Mechanical BCs

- Dirichlet BCs : $\vec{u} = \vec{u}_e$ on $\Gamma_D$
- Neumann BCs : $\nabla \vec{u} \cdot \vec{n} = \vec{h}_e$ on $\Gamma_N$

⇝ $\underbrace{\underbrace{[c] : \nabla \vec{u}}_{[\sigma]} \cdot \vec{n}}_{\text{normal tractions on boundary } \Gamma_N} = \vec{f}_e$ on $\Gamma_N$

## COMSOL nodes (boundary level)

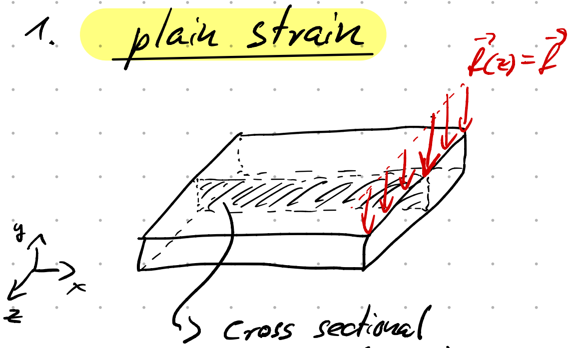
- free : $([c] : \nabla \vec{u}) \cdot \vec{n} = \vec{0}$ ⇝ no tractions/forces at boundary
  ↳ homogeneous NBC
- boundary loads : $[\sigma] \cdot \vec{n} = ([c] : \nabla \vec{u}) \cdot \vec{n} = \vec{t}_N$
  ↳ inhomogeous NBC
- fixed constraint : $\vec{u} = \vec{0}$ ⇝ homogeneous DBC
- prescribed displacement : $\vec{u} = \vec{u}_e$ ⇝ inhomogeneous DBC
- symmetry : $\vec{u} \cdot \vec{n} = 0$

## COMSOL nodes (point level)

- point load : $([c] : \nabla \vec{u}) \cdot \vec{n} = \vec{t}_N$ on selected node

# Mechanics: 2D approximations

## 1. plain strain




↳ cross sectional area $A(z=z_0)$

assumptions: • $\frac{\partial \vec{u}}{\partial z} = 0$ and

• $u_z = 0$.

$\Rightarrow$ all $A(z)$ are the same

↳ with same loads ⇝ line loads

↳ ~~no z-component~~!

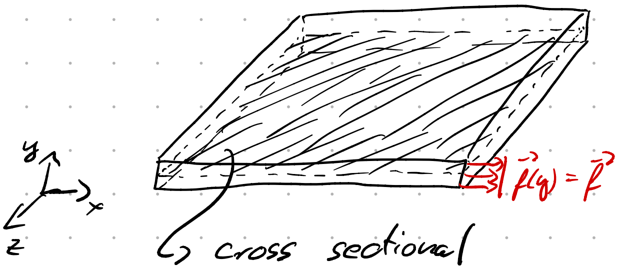
↳ infinite in z-direction

$$\Rightarrow \quad S_{zx} = S_{zy} = S_{zz} = 0$$

rule of thumb for required geometry:

body size in z-direction $\gg$ other dimensions.

## 2. plain stress




↳ cross sectional area $A(y=y_0)$

assumptions: • $\frac{\partial \vec{u}}{\partial y} = 0$ and

• traction along y-dimension is zero: $\sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0$.

$\Rightarrow$ infinitely thin in y-direction so that no forces in y are supported.

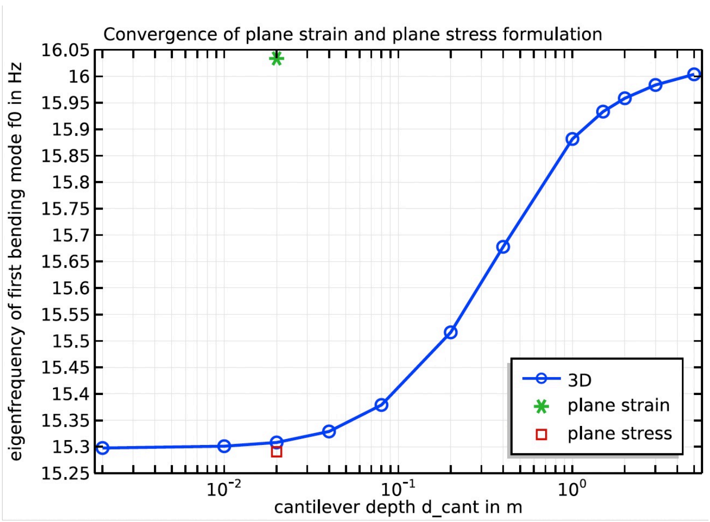
→ there is basically only one representative $A(y_0)$

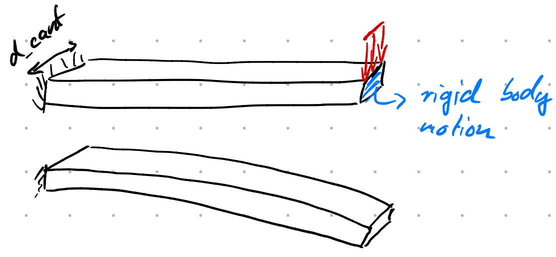
→ loading is independent of y and it has no y-component

rule of thumb for required geometry:

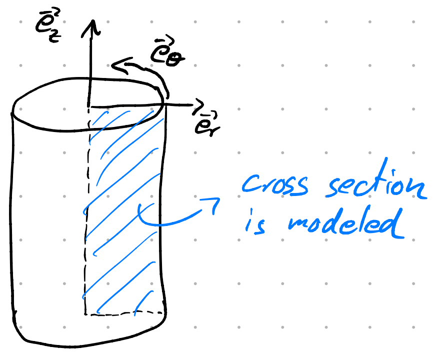
body size in y-direction $\approx$ or $\ll$ other dimensions.

## Comparison to 3D



# Mechanical field: axi-symmetry




- cylindrical coordinate system (COS):

  radial: $\vec{e}_r = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$

  axial: $\vec{e}_z$

  circumferential: $\vec{e}_\theta = -\sin\theta\, \vec{e}_x + \cos\theta\, \vec{e}_y$

$$\nabla := \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \rightsquigarrow \quad \nabla_{cyl} := \begin{pmatrix} \partial/\partial r \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

assumptions:
- mechanical field does not depend on $\theta$ $\rightsquigarrow$ $\frac{\partial u_i}{\partial \theta} = 0$
- $u_\theta = 0$

requires:
1. torque is zero
2. load does not depend on $\theta$
3. load is not acting in $\vec{e}_\theta$

we obtain $\quad s_{r\theta} = 0, \quad s_{z\theta} = 0,$

but in general not $s_{\theta\theta} \neq 0,$

because:

$$\begin{aligned}
s_{\theta\theta} &= \frac{1}{r}\frac{\partial}{\partial\theta}\vec{u}\cdot\vec{e}_\theta \\
&= \frac{1}{r}\frac{\partial}{\partial\theta}\left(u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_z\vec{e}_z\right)\cdot\vec{e}_\theta \\
&= \frac{1}{r}\left(\cancel{\frac{\partial u_r}{\partial\theta}}^{0}\vec{e}_r + u_r\frac{\partial\vec{e}_r}{\partial\theta} + \cancel{\frac{\partial u_\theta}{\partial\theta}}^{0}\vec{e}_\theta + \cancel{u_\theta}^{0}\frac{\partial\vec{e}_\theta}{\partial\theta} + \cancel{\frac{\partial u_z}{\partial\theta}}^{0}\vec{e}_z + u_z\cancel{\frac{\partial\vec{e}_z}{\partial\theta}}^{0}\right)\cdot\vec{e}_\theta \\
&= \frac{u_r}{r}\frac{\partial\vec{e}_r}{\partial\theta}\cdot\vec{e}_\theta .
\end{aligned}$$

observe that $\frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta$

$$\Rightarrow \quad s_{\theta\theta} = \frac{u_r}{r}\,\vec{e}_\theta\cdot\vec{e}_\theta = \frac{u_r}{r}$$

which is in general not equal to zero.

REMEMBER: For curvilinear COS the dependence of $\vec{e}_i$ on the coordinates $x_j$ needs to be considered!

REMEMBER: The $\nabla$-operator needs to consider the COS!

# Quasi-static Magnetic Field

- assume $\frac{\partial \vec{D}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \vec{H} = \vec{J}$ (Ampère's law)
- magnetic vector potential: $\vec{B} = \nabla \times \vec{A}$
- Faradays law: $\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$
- material laws: $\vec{J} = \gamma \vec{E}$
  $\vec{H} = \nu \nabla \times \vec{A}$
- nonmoving setup
- irrotational currents prescribed

$\Rightarrow$ quasi-static magnetic PDE:

$$\gamma \frac{\partial \vec{A}}{\partial t} + \nabla \times \nu \nabla \times \vec{A} = \vec{J}_i$$

<u>prove</u>: $\vec{B} = \nabla \times \vec{A} \quad \Rightarrow \quad \nabla \cdot \vec{B} = 0 \quad \forall \vec{A}(x,y,z)$

$$\nabla \cdot (\nabla \times \vec{A}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \left[ \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \right] = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{bmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{bmatrix}$$

$$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} + \cancel{\frac{\partial^2 A_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_y}{\partial z \partial x}} - \cancel{\frac{\partial^2 A_x}{\partial z \partial y}} \equiv 0 \qquad \square$$

$\Rightarrow$ such a $\vec{B}$ is purely solenoidal
$\Rightarrow$ implicitly satisfies Maxwell's law

<u>prove</u>: $\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$ is not unique

$\vec{A}$ satisfies $\nabla \cdot (\nabla \times \vec{A}) = 0$.

choose $\vec{A}^* = \vec{A} + \nabla \psi$, where $\psi$ is a scalar potential

$$\vec{B} = \nabla \times \vec{A}^* = \nabla \times (\vec{A} + \nabla \psi) = \nabla \times \vec{A} + \underbrace{\cancel{\nabla \times \nabla \psi}}^{0}$$

$\Rightarrow$ $\vec{A}^*$ is also a solution

$\rightsquigarrow$ make it unique with additional constraint
$\hookrightarrow$ usually Coloumb gauge: $\nabla \cdot \vec{A} \stackrel{!}{=} 0$

<u>prove:</u> $\nabla\cdot\vec{A} = 0$ automatically in 2D

$\rightarrow$ 2D: $B_z = 0$ and $\frac{\partial \vec{B}}{\partial z} = \vec{0}$ $\rightsquigarrow$ $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix}$

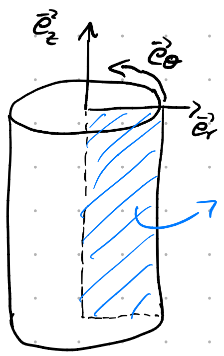
$$\nabla\times\vec{A} = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \cancel{\frac{\partial A_y}{\partial z}}^{\to 0} \\ \cancel{\frac{\partial A_x}{\partial z}}^{\to 0} - \frac{\partial A_z}{\partial x} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial A_z}{\partial y} \\ -\frac{\partial A_z}{\partial x} \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ $\vec{B}$ in 2D depends only on $A_z$: $\vec{A} = \begin{pmatrix} 0 \\ 0 \\ A_z \end{pmatrix}$

$$\nabla\cdot\vec{A} = \frac{\partial A_z}{\partial z} \overset{!}{=} 0 \quad \Rightarrow \quad A_z = A_z(x,y)$$

REMEMBER: in 2D the magnetic vector potential is normal to the simulation plane and it does not depend on this out of plane coordinate.
$\rightsquigarrow$ such $\vec{A}$ is unique and satisfies $\nabla\cdot\nabla\times\vec{A} = 0$.

<u>prove:</u> $\nabla\cdot\vec{A} = 0$ automatically in 2D axi-symmetric case




axial symmetry: $B_\theta = 0$, $\frac{\partial \vec{B}}{\partial \theta} = 0$

$$\nabla_{cyl} = \begin{pmatrix} \partial/\partial r \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \partial/\partial z \end{pmatrix}$$

$$\vec{B} = \nabla_{cyl}\times\vec{A} = \begin{pmatrix} \partial/\partial r \\ \frac{1}{r}\frac{\partial}{\partial\theta} \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix}$$

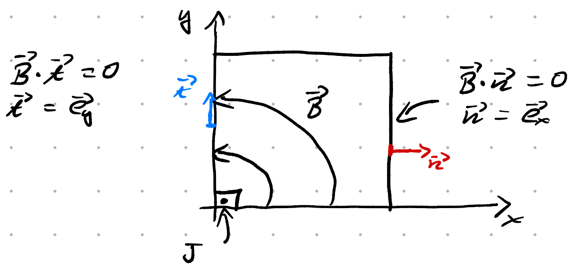
$$= \begin{pmatrix} \frac{1}{r}\cancel{\frac{\partial A_z}{\partial\theta}}^{\to 0} - \frac{\partial A_\theta}{\partial z} \\ \cancel{\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}}^{\to 0} \\ \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\cancel{\frac{\partial A_r}{\partial\theta}}^{\to 0} \end{pmatrix} = \begin{pmatrix} -\frac{\partial A_\theta}{\partial z} \\ 0 \\ \frac{\partial A_\theta}{\partial r} \end{pmatrix}$$

$\rightsquigarrow$ depends only on $A_\theta$! $\Rightarrow$ $\vec{A} = \begin{pmatrix} 0 \\ A_\theta \\ 0 \end{pmatrix}$

impose Coulomb gauge: $\nabla_{cyl}\cdot\vec{A} = \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} = 0$

$\Rightarrow$ $A_\theta = A_\theta(r,z)$ is plain

# Magnetic boundary conditions




1. $\vec{B}\cdot\vec{n} = (\nabla\times\vec{A})\cdot\vec{n} \overset{!}{=} 0$

$$\begin{pmatrix} \frac{\partial A_z}{\partial y} \\ \frac{\partial A_z}{\partial x} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\partial A_z}{\partial y} \overset{!}{=} 0$$

$\leadsto$ $A_z = A_z(x)$ does not depend on y.
$\leadsto$ $A_z$ is constant along y

$\Rightarrow$ Dirichlet BC.
$\leadsto$ flux is parallel

2. $\vec{B}\cdot\vec{t} = (\nabla\times\vec{A})\cdot\vec{t} \overset{!}{=} 0$

$$\begin{pmatrix} \frac{\partial A_z}{\partial y} \\ \frac{\partial A_z}{\partial x} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\partial A_z}{\partial x} = -\nabla A_z\cdot\vec{n} = 0$$

$\Rightarrow$ homogeneous Neumann BC
$\leadsto$ flux is normal

## COMSOL node

### Domain contributions

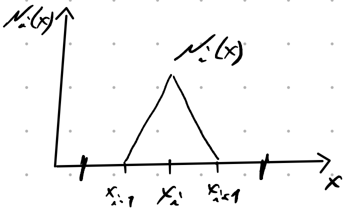
- Ampère's law: adds the magnetic PDE and constitutive rel.
- external current density: imposes an externally generated current density. $\leadsto$ loading term
- velocity (Lorentz term): add a current density of:
  $J_L = \sigma\, \vec{v}\times\vec{B}$

### Boundary contributions

- magnetic insulation: tangential component of vector potential is set to zero: $\vec{n}\times\vec{A}=0$.
  $\leadsto$ default BC. $\leadsto$ homogeneous DBC
  $\leadsto$ flux parallel on boundary
- perfect magnetic conductor: $\vec{n}\times\vec{H}=0$
  $\leadsto$ $\vec{n}\times(\frac{1}{\mu}\nabla\times\vec{A})=0$ $\leadsto$ scaled homogeneous NBC
  $\leadsto$ flux normal on boundary

# Handling irregular meshes

Motivation: until now only equidistant 1D meshes

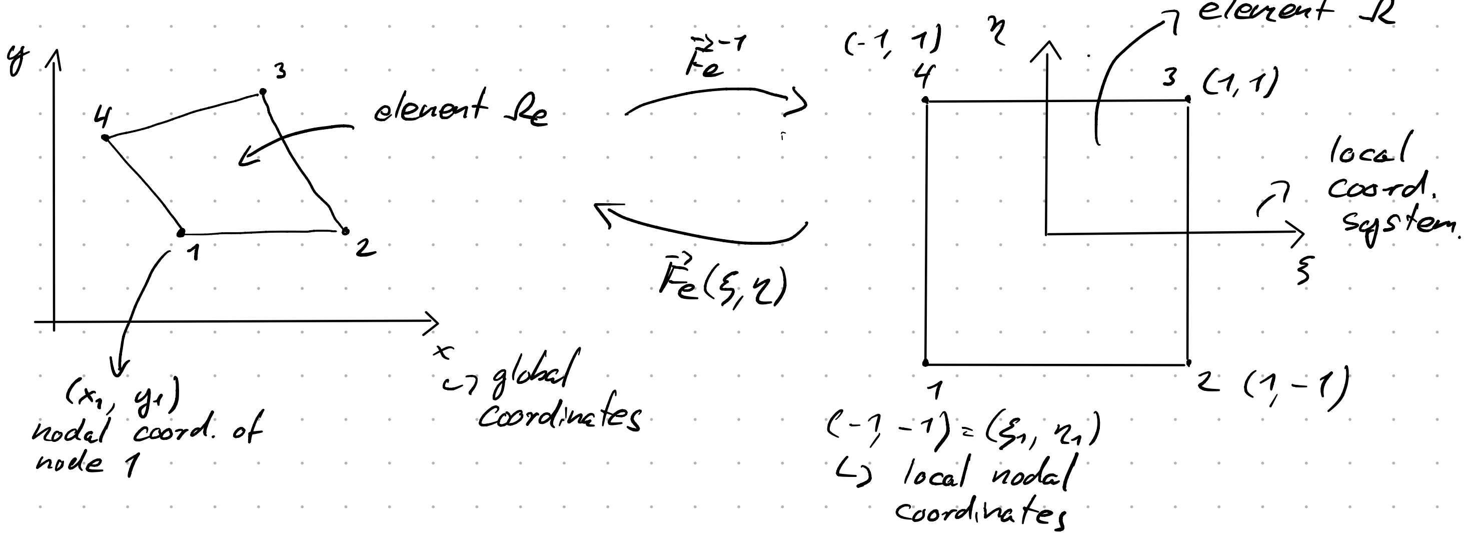



-> what about arbitrary meshes in 1D, 2D and 3D?
↳ automatic computation

solution approach:

1) define reference elements
   -> define shape functions $N_i$ and their derivatives on these reference elements
   -> hard code the ref. shape functions and derivatives in the FE-code.
2) transform actual elements / shape functions onto ref. elements
   -> this can be done automatically!

## Isoparametric mapping: example in 2D




How to perform computations on the local coordinates:

1) <u>determine the mapping</u>: do by hand and hardcode results

- introduce the map $\vec{F}_e : \hat{\Omega} \to \Omega_e$

$$\vec{F}_e = \begin{cases} x^e(\xi,\eta) = \sum_{a=1}^{N_a} N_a(\xi,\eta)\, x_a^e \\ y^e(\xi,\eta) = \sum_{a=1}^{N_a} N_a(\xi,\eta)\, y_a^e \end{cases} \qquad (1)$$

with the same interpolation functions $N_a$ as for the unknown function $u^h(x,y) = \sum N_i(x,y)\, u_i$. These elements are called "isoparametric" elements".

- compute the basis functions $N_a$ by:
  -> choose $x^e(\xi,\eta) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi\eta + \alpha_3 \eta$
  ($N_a$ is polynomial of 1st order)
  $y^e(\xi,\eta) = \beta_0 + \beta_1 \xi + \beta_2 \xi\eta + \beta_3 \eta$

  -> require $x^e(\xi_a, \eta_a) = x_a^e$
  $y^e(\xi_a, \eta_a) = y_a^e$

  -> yields 8 equations to determine the 8 unknowns $\alpha_a$, $\beta_a$.
  -> comparing to 1, we obtain:

  nodal coordinates

  $$N_a(\xi,\eta) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$

- $$\hat{\nabla} = \begin{pmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{pmatrix} \rightsquigarrow \hat{\nabla} N_a(\xi,\eta) = \frac{1}{4}\begin{bmatrix} \xi_a(1+\eta_a\eta) \\ \eta_a(1+\xi_a\xi) \end{bmatrix}$$

  -> hard code in FE-code. <-

## 2) transform elements onto ref. elements

-> resulting expressions are valid for all elements (independent of size and spape!)
-> the expressions are computed numerically for each element

transform by
-> performing a change of variables $(x,y) \mapsto (\xi,\eta)$ using the map $\vec{F}_e$.

Transformation of:

- an integral: $$\int_{\Omega_e} f(x,y)\, dx\,dy = \int_{\hat{\Omega}} \hat{f}(\xi,\eta) \underbrace{\det J_e \, d\xi\, d\eta}_{dx\,dy}$$

  where Jabian matrix is

  $$J_e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

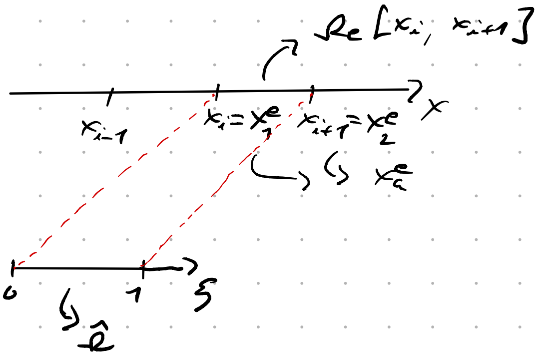
- the nabla operator $\nabla$: $\nabla \to J_e^{-T}\hat{\nabla}$
  ↳ this transforms derivatives w.r.t. the global coord. to derivatives w.r.t. the local coord.

- replace dependences on $x$ and $y$ by dependences on $\xi$ and $\eta$.

  $$f(x,y) \mapsto \hat{f}^e(\xi,\eta) = f(x^e(\xi,\eta), y^e(\xi,\eta))$$

  ↓
  expression changes for every element!
  ↳ but can be obtained automatically

# Isoparametric mapping in 1D




1) compute the mapping $F_e$:

- $F_e = x(\xi) = \sum_{a=1}^{2} N_a(\xi)\, x_a^e$

  ↳ isoparametric ansatz

  $\overset{!}{=} \alpha_0 + \alpha_1 \xi$

- require

$$x(\xi_1) = \alpha_0 + \cancel{\alpha_1 \xi_1}^{\,0} = \alpha_0 = x_1^e$$

$$x(\xi_2) = \alpha_0 + \alpha_1 \xi_2 = \alpha_0 + \alpha_1 = x_2^e$$

$$\Rightarrow \alpha_1 = x_2^e - x_1^e = h_e$$

$$\Rightarrow \quad x(\xi) = x_1^e + (x_2^e - x_1^e)\xi = (1-\xi)\,x_1^e + \xi(x_2^e)$$

$$\Rightarrow \begin{cases} N_1(\xi) = 1 - \xi \\ N_2(\xi) = \xi \end{cases} \quad \Rightarrow \begin{cases} \frac{\partial}{\partial \xi} N_1(\xi) = -1 \\ \frac{\partial}{\partial \xi} N_2(\xi) = 1 \end{cases}$$

2) transform the element stiffness and force integral representations to local coordinates

- element stiffness matrix: $k_{ab}^e = k \int_{\Omega_e} N_a'(x)\, N_b'(x)\, dx$

Jacobi matrix $\quad J_e = \frac{\partial x(\xi)}{\partial \xi} = x_2^e - x_1^e$

transform derivative: $\quad \frac{\partial}{\partial x} \mapsto J_e^{-T} \frac{\partial}{\partial \xi}$

transform $dx$: $\quad dx \mapsto \det J_e \, d\xi$

transform domain: $\quad \Omega_e \mapsto \hat{\Omega} = [0, 1]$

the shape functions depending on $\xi$ are known from above.

$$\Rightarrow \quad k_{ab}^e = k \int_0^1 J_e^{-T} \frac{\partial}{\partial \xi} N_a(\xi)\, J_e^{-T} \frac{\partial}{\partial \xi} N_b(\xi)\, \det J_e \, d\xi$$

$$= \underbrace{k\,(J_e^{-T})^2 \det J_e}\int_0^1 N_a'(\xi)\, N_b'(\xi)\, d\xi$$

↳ always the same and known, dash is $\partial/\partial\xi$ in this case.

↳ integration is always from 0 to 1!

↳ does not depend on the coordinate $\xi$, but is different for every element $e$.

computation in local/reference coordinates!

- element force vector: $f_a^e = \int_{\Omega_e} f(x) N_a(x)\, dx$

$$N_a(x) \mapsto N_a(\xi)$$
$$f(x) \mapsto \hat{f}^e(\xi) = f(x(\xi)) = f(\underbrace{(1-\xi)}_{N_1^e(\xi)} x_1^e + \underbrace{(\xi)}_{N_2^e(\xi)} x_2^e)$$
$$dx \mapsto \det J_e \, d\xi$$
$$\Omega_e \mapsto \hat{\Omega} = [0, 1]$$

$$\Rightarrow \quad f_a^e = \int_0^1 \hat{f}^e(\xi) N_a(\xi) \det J_e \, d\xi$$
$$= \det J_e \int_0^1 \hat{f}^e(\xi) N_a(\xi) \, d\xi$$

$N_a(\xi)$: ↳ known (hard coded)

$\hat{f}^e(\xi)$: ↳ implemented in "toLocalCoordinates()"

↓ computation in terms of local/ref. coordinates

## Notes on FE-implementation

- hard code $\begin{cases} N_1(\xi) \\ N_2(\xi) \end{cases}$ and $\begin{cases} \frac{\partial}{\partial \xi} N_1(\xi) \\ \frac{\partial}{\partial \xi} N_2(\xi) \end{cases}$
- when assembling the global matrices/vectors and looping over the elements:
  - → compute $J_e$, $\det J_e$, $J_e^{-T}$ and $\hat{f}^e$, as required
  - → define the integrand as an anonymous function
  - → perform a numerical integration

# Acoustics

- special case of mechanics
  ↳ less DoF
- standard problem types:
  - radiation
  - scattering
  - field in interior space
  - transducer problems

- Acoustic field (linearized)
  - mass conservation: $\varrho_0 \nabla \cdot \vec{v}' = -\frac{\partial \varrho'}{\partial t}$ (continuity eq.)
    total density given by $\underbrace{\varrho_0}_{\text{mean}} + \underbrace{\varrho'}_{\text{change}}$
    ⇝ similar for all quantities!
- conservation of momentum: $\varrho_0 \frac{\partial \vec{v}'}{\partial t} = -\nabla p'$ (Euler eq.)
  $p'$: acoustic pressure
- state equation: Taylor series: $\varrho' = \cancel{\varrho_0'}^{\,0} + \underbrace{\left.\frac{\partial \varrho'}{\partial p'}\right|_{\varrho_0'}}_{=:\frac{1}{c^2}} p' + \mathcal{O}(p'^2)$
  there is no mean in the dashed quantities!
  $\Rightarrow \varrho' = \frac{1}{c^2} p'$
  - 5 scalar eq. for 5 scalar unknows from: $\varrho'$, $p'$, $\vec{v}'$.
  - merge eqs. into "wave equation":
    $$\vec{\nabla} \cdot \vec{\nabla} p' = \Delta p' = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p'$$

## Acoustic scalar potential

- particle velocity $\vec{v}'$ is irrotational:
  curl of momentum eq.: $\varrho_0 \frac{\partial}{\partial t} \nabla \times \vec{v}' = -\nabla \times \nabla p' = \vec{0}$
  ↳ will automatically be fulfilled if we chose $\vec{v}' = -\nabla \psi$
  ↳ acustic scalar potential $\psi$
- relation to pressure: insert into momentum eq.:
  $-\varrho_0 \frac{\partial}{\partial t} \nabla \psi = -\nabla p' \Rightarrow p' = \varrho_0 \frac{\partial \psi}{\partial t}$
- insert the above into the wave eq.: $\Delta \varrho_0 \frac{\partial \psi}{\partial t} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varrho_0 \frac{\partial \psi}{\partial t}$
  $\Rightarrow \Delta \psi = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi$ → solves same wave eq. !

- REMEMBER:
    - acoustic field is irrotational
    - only longitudinal waves propagate
    - simplification for fluids (gases and liquids)

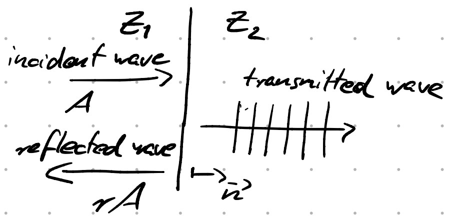
## Acoustic BCs

- sound soft: $p' = 0$ ⤳ homogeneous Dirichlet BC
- sound hard: $\vec{v}' \cdot \vec{n} = 0$ ⤳ homogeneous Neumann BC
- impedance BC: $\dfrac{p'}{\vec{v}' \cdot \vec{n}} = Z_0$

(in pressure acoustics)

## Total reflections from domain boundaries

reflection coefficient: $r = \dfrac{Z_2 - Z_1}{Z_2 + Z_1}$



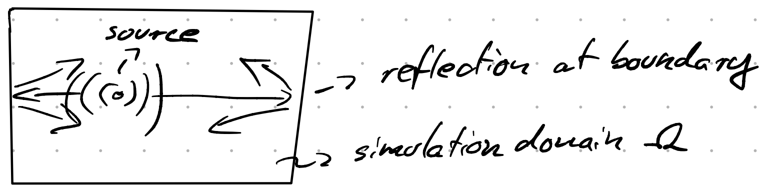

→ total reflection if $|r| = 1$

general definition of impedance: $Z = \dfrac{p'}{\vec{v}' \cdot \vec{n}}$

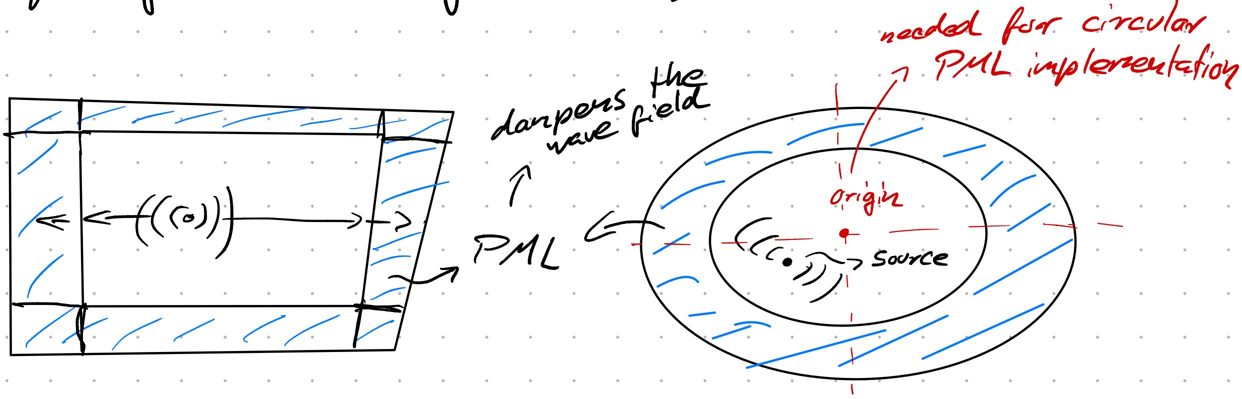
plane wave impedance: $Z = \rho_0 c$

- sound soft: $p_2' = 0 \implies Z_2 = 0 \implies r = -1$
- sound hard: $\vec{v}_2' \cdot \vec{n} \to 0 \implies Z_2 \to \infty \implies r = 1$

⤳ REMEMBER: Both sound hard and sound soft BCs lead to total reflection of sound at the boundary.




⤳ How to implement radiation problems?
- → time domain with big domain
- → radiation/absorbing BC
- → perfectly matched layer (PML)



# COMSOL: pressure acoustics, frequency domain

- dependent variable is $p'$
- ansatz $\quad p'(\vec{x}, t) = p(\vec{x}) e^{j\omega t}, \quad p(\vec{x})$ is complex valued

  insert into wave eq.:

$$\Delta p(\vec{x}) \cancel{e^{j\omega t}} = \frac{1}{c} (j\omega)^2 p(\vec{x}) \cancel{e^{j\omega t}}$$

  introduce wave number $\quad k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$

$$\Rightarrow \quad \Delta p + k^2 p = 0 \qquad \text{(homogeneous Helmholz eq.)}$$

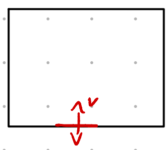
  ⇝ sources can be added to the RHS.

## COMSOL domain contributions

- pressure acoustics: adds Helmholz eq.

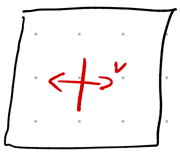
## COMSOL boundary contributions

- sound hard boundary: $\frac{1}{\rho_0} \vec{\nabla} p \cdot \vec{n} = 0 \quad$ (default BC)
  - ⇝ momentum eq.: $\frac{1}{\rho_0} \vec{\nabla} p \cdot \vec{n} = - \frac{\partial}{\partial t} \vec{v} \cdot \vec{n}$
  - ⇝ is setting the normal acceleration to zero
  - ⇝ hence $\vec{v} \cdot \vec{n} = 0$ for harmonic analysis
  - ⇝ this is a homogeneous Neumann BC
- symmetry BC = sound hard BC
- sound soft boundary: $\quad p = 0$
  - ⇝ homogeneous Dirichlet BC
- plane wave radiation ≈ absorbing BC
- normal velocity:




- "interior" versions of the above:

  e.g. interior normal vel.:



# Exercise sheet 2

## Problem 1

weak form: $$\int_\Omega v'u' + c\,v\,u' \,dx = \int_\Omega v f \,dx \qquad (1)$$

$$\text{DBC}: \quad u(a) = g_a \,, \quad u(b) = g_b$$

ansatz: $$v^h(x) = \sum_{i=2}^{N-1} N_i(x)\, v_i$$

$$u^h(x) = \sum_{j=2}^{N-1} N_j(x)\, u_j + N_1(x)\, g_a + N_N(x)\, g_b$$

1. put $v^h$ and $u^h$ into (1):

$$\int_\Omega \Big(\sum_{i=2}^{N-1} N_i' v_i\Big)\Big(\sum_{j=2}^{N-1} N_j' u_j + N_1' g_a + N_N' g_b\Big)\,dx$$
$$+ \int_\Omega c\Big(\sum_{i=2}^{N-1} N_i v_i\Big)\Big(\sum_{j=2}^{N-1} N_j' u_j + N_1' g_a + N_N' g_b\Big)\,dx$$
$$= \int_\Omega \Big(\sum_{i=2}^{N-1} N_i v_i\Big) f \,dx$$

2. Move sums and $u_j, v_i$ out of the integrals:

$$\sum_{i=2}^{N-1} v_i \Big[\sum_{j=2}^{N-1} \int_\Omega N_i' N_j' \,dx\, u_j + \int_\Omega N_i' N_1' \,dx\, g_a + \int_\Omega N_i' N_N' \,dx\, g_b\Big]$$
$$+ \sum_{i=2}^{N-1} v_i \Big[\sum_{j=2}^{N-1} \int_\Omega c N_i N_j' \,dx\, u_j + \int_\Omega c N_i N_1' \,dx\, g_a + \int_\Omega c N_i N_N' \, g_b\Big]$$
$$= \sum_{i=2}^{N-1} v_i \Big[\int_\Omega N_i f \,dx\Big]$$

3. $v_i$ are arbitrary

$$\sum_{j=2}^{N-1} \underbrace{\int_\Omega N_i' N_j' \,dx}^{(N-2)\times(N-2)\text{ coefficients}}\, u_j + \int_\Omega N_i' N_1' \,dx\, g_a + \int_\Omega N_i' N_N' \,dx\, g_b$$
$$+ \sum_{j=2}^{N-1} \int_\Omega c N_i N_j' \,dx\, u_j + \int_\Omega c N_i N_1' \,dx\, g_a + \int_\Omega c N_i N_N' \,dx\, g_b$$
$$= \int_\Omega N_i f \,dx$$

4. reorder terms: move all known quatities to the right

$$\sum_{j=2}^{N-1} \underbrace{\int_\Omega N_i' N_j' + c N_i N_j' \, dx}_{S_{ij}} \, u_j = \underbrace{\int_\Omega N_i f \, dx}_{f_i} +$$

$$- \underbrace{\int_\Omega N_i' N_1' + c N_i N_1' \, dx}_{S_{i1}} \, g_a - \underbrace{\int_\Omega N_i' N_N' + c N_i N_N' \, dx}_{S_{iN}} \, g_b$$

5. symbolic notation: $\underline{\underline{S}} \cdot \underline{u} = \underline{f} - \underline{S}_1 g_a - \underline{S}_N g_b$

# Problem 2

(1) weak: $\int_\Omega v' u' \, dx = \int_\Omega v f \, dx + v(b) h_b$ on $x \in \Omega = (a,b)$

$\rightsquigarrow v^h(b) = \sum_{i=2}^{N} \delta_{iN} v_i = v_N$

with $\delta_{iN} = \begin{cases} 1 & i = N \\ 0 & i \neq N \end{cases}$

DBC: $u(a) = g_a$

FE ansatz:
$$v^h(x) = \sum_{i=2}^{N} N_i(x) v_i + \cancel{v_1}^{\,0} N_1(x)$$
$$u^h(x) = \sum_{j=2}^{N} N_j(x) u_j + N_1 g_a$$

1. plug into (1):

$$\int_a^b \sum_{i=2}^{N} N_i' v_i \left( \sum_{j=2}^{N} N_j' u_j + N_1' g_a \right) dx = \int_a^b \left( \sum_{i=2}^{N} N_i v_i \right) f \, dx + \sum_{i=2}^{N} \delta_{iN} v_i h_b$$

2. move sums and $u_j$, $v_i$ out of the integrals:

$$\sum_{i=2}^{N} v_i \left[ \sum_{j=2}^{N} \int_a^b N_i' N_j' \, dx \, u_j + \int_a^b N_i' N_1' \, dx \, g_a \right] = \sum_{i=2}^{N} v_i \left[ \int_a^b N_i f \, dx + \delta_{iN} h_b \right]$$

3. $v_i$ can be arbitrary $\rightsquigarrow$ remove from eq.:

$$\sum_{j=2}^{N} \int_a^b N_i' N_j' \, dx \, u_j + \int_a^b N_i' N_1' \, dx \, g_a = \int_a^b N_i f \, dx + \delta_{iN} h_b$$

4. move known terms to the right:

node 'N' is at the Neumann boundary

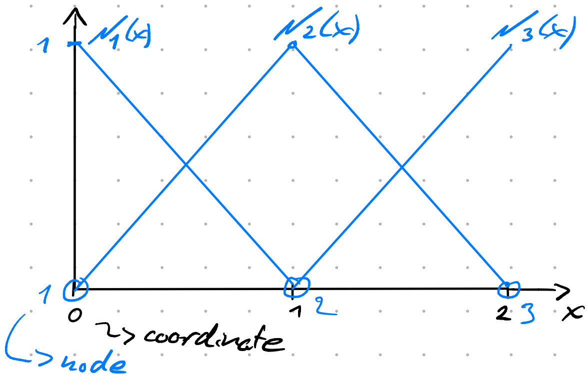
$$\sum_{j=2}^{N} \underbrace{\int_a^b N_i' N_j' \, dx}_{S_{ij}} u_j = \underbrace{\int_a^b N_i f \, dx}_{f_i} + \underbrace{\delta_{iN} h_b}_{N_i} - \underbrace{\int_a^b N_i' N_1' \, dx}_{S_{i1}} g_a$$

$N_i \downarrow$ Neumann contribution

$S_{i1} \downarrow$ Dirichlet contribution

# Exercise sheet 3

PDE: $u'' + 6u = 2$ on $x \in \Omega = [0,2]$
BC: $u(0) = 1$ , $u(2) = 2$




$$N_1(x) = \begin{cases} 1-x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2-x & x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

## Problem 1

- include BCs in FE-ansatz and solve:

1. Derive the weak form:

$$\int_0^2 v u'' + 6 v u \, dx = \int_0^2 v \, 2 \, dx$$

$$\overset{0 \to DBC}{\cancel{[v u']_0^2}} - \int_0^2 v' u' \, dx + \int_0^2 6 v u \, dx = \int_0^2 2 v \, dx$$

2. Ansatz for Galerkin discretization:

$$v \approx v^h(x) = \sum_i v_i N_i(x) = v_2 N_2(x)$$
$$u \approx u^h(x) = \sum_j u_j N_j(x) = 1 N_1(x) + u_2 N_2(x) + 2 N_3(x)$$

3. insert into week form:

$$\underbrace{-\int_0^2 \cancel{v_2} N_2' (N_1' + u_2 N_2' + 2 N_3') \, dx}_{A} + \underbrace{\int_0^2 6 \cancel{v_2} N_2 (N_1 + u_2 N_2 + 2 N_3) \, dx}_{B}$$
$$= \underbrace{\int_0^2 2 \cancel{v_2} N_2 \, dx}_{F}$$

$$A = -\int_0^1 1(-1 + u_2 1 + 2 \cdot 0) dx - \int_1^2 (-1)(0 + u_2(-1) + 2(+1)) dx$$
$$= -\int_0^1 u_2 - 1 \, dx - \int_1^2 u_2 - 2 \, dx$$
$$= -[(u_2 - 1)x]_0^1 - [(u_2 - 2)x]_1^2$$
$$= -(u_2 - 1) - (u_2 - 2) = -2u_2 + 3$$

$$B = \underbrace{\int_0^2 6 \cancel{v_2} N_2 (N_1 + u_2 N_2 + 2 N_3)\, dx}_{B}$$

$$N_1(x) = \begin{cases} 1-x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2-x & x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x-1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$B = \int_0^1 6x(1-x + u_2(x) + 2 \cdot 0\, dx + \int_1^2 6(2-x)(0 + u_2(2-x) + 2(x-1)\, dx$$

$$\vdots$$

$$= 3 + 4u_2$$

$$F = \underbrace{\int_0^2 2 \cancel{v_2} N_2\, dx}_{F}$$

$$F = \int_0^1 2x\, dx + \int_1^2 2(2-x)\, dx$$

$$= 2$$

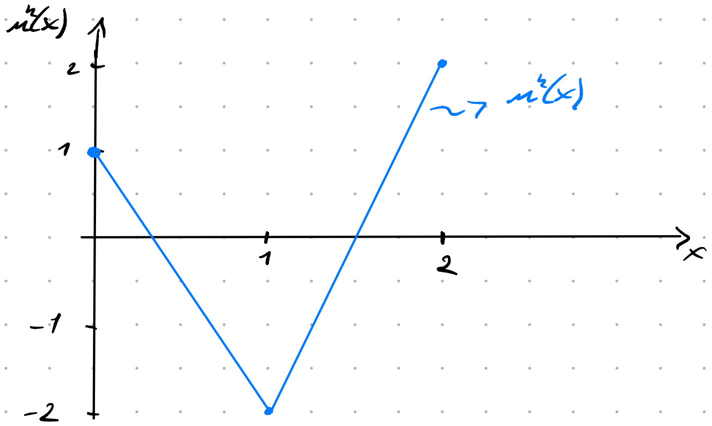
- plug back into equation: $A + B = F$

$$-2u_2 + 3 + 3 + 4u_2 = 2$$

$$\Rightarrow 2u_2 = -4$$

$$\Rightarrow u_2 = -2 \quad , \quad \text{DBC values: } u_1 = 1, \; u_3 = 2$$

- plot solution:



## Problem 2 (Note: you can ignore the blue parts)

- first derive Galerkin discretization, then incorporate BCs.

ansatz: $\quad v^h = \sum_{i=1}^{3} v_i N_i(x) \quad , \quad u^h = \sum_{j=1}^{3} u_j N_j(x)$

weak form: $\quad -\int_0^2 v' u' \, dx + \int_0^2 6 v u \, dx = \int_0^2 2 v \, dx - [v u']_0^2$

expressions at boundary: $\quad u^h(2) = \sum_j u_j \delta_{j3}, \; u^h(0) = \sum_j u_j \delta_{j1}, \quad v^h(2) = \sum_i v_i \delta_{i3}, \quad v^h(0) = \sum_i v_i \delta_{i1}$

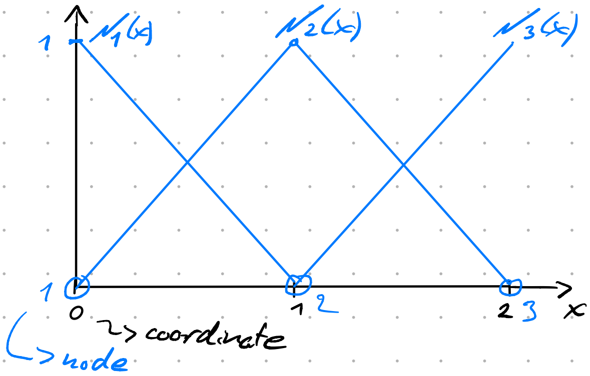
$\Rightarrow u^{h\prime}(0) = \frac{u_2 - u_1}{1} = \sum_j u_j (\delta_{j2} - \delta_{j1}), \quad u^{h\prime}(2) = \frac{u_3 - u_2}{1} = \sum_j u_j (\delta_{j3} - \delta_{j2}) \rightsquigarrow$ slope with $x_{i+1} - x_i = 1$

insert ansatz into weak form

$$\sum_i v_i \sum_j \left[ \int_0^2 - N_i' N_j' \, dx + \int_0^2 6 N_i N_j \, dx \right] u_j + \sum_i v_i \sum_j \left[ \delta_{i3}(\delta_{j3} - \delta_{j2}) - \delta_{i1}(\delta_{j2} - \delta_{j1}) \right] u_j = \sum_i v_i \int_0^2 2 N_i \, dx$$

test functions $v^h(x)$ are arbitrary $\rightsquigarrow$ eliminate $v_i$ and $\sum_i$:

$$\sum_j \Big[ \underbrace{\int_0^2 - N_i' N_j' \, dx}_{A_{ij}} + \underbrace{\int_0^2 6 N_i N_j \, dx}_{B_{ij}} + \underbrace{\delta_{i3}(\delta_{j3} - \delta_{j2}) - \delta_{i1}(\delta_{j2} - \delta_{j1})}_{C_{ij}} \Big] u_j = \underbrace{\int_0^2 2 N_i \, dx}_{F_i}$$




$$N_1(x) = \begin{cases} 1 - x & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$N_2(x) = \begin{cases} x & \text{for } x \in [0,1] \\ 2 - x & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$N_3(x) = \begin{cases} x - 1 & \text{for } x \in [1,2] \\ 0 & \text{else} \end{cases}$$

$$\left.\begin{aligned} A_{11} &= \int_0^1 -(-1)(-1) \, dx + \int_1^2 0 \, dx = -1 \\ A_{22} &= \int_0^1 -(1)(1) \, dx + \int_1^2 -(-1)(-1) \, dx = -2 \\ A_{12} &= \int_0^1 -(-1)(1) \, dx + \int_1^2 -(0)(-1) \, dx = 1 \end{aligned}\right\} \begin{matrix}\text{sym!} \\ \text{sparse!}\end{matrix} \rightsquigarrow A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B_{11} = 6 \int_0^1 (1-x)(1-x) \, dx = 6 \int_0^1 1 - 2x + x^2 \, dx = 6 \left[ x - 2x^2/2 + x^3/3 \right]_0^1 = 2$$

$$B_{12} = 6 \int_0^1 (1-x)(x) \, dx + 6 \int_1^2 (0)(2-x) \, dx = 6 \int_0^1 (x - x^2) \, dx = 6 \left[ x^2/2 - x^3/3 \right]_0^1 = 1$$

$$B_{22} = 6 \int_0^1 (x^2) \, dx + 6 \int_1^2 (2-x)(2-x) \, dx = 6 \int_0^1 x^2 \, dx + 6 \int_1^2 (4 - 4x + x^2) \, dx$$

$$= 6 \left[ x^3/3 \right]_0^1 + 6 \left[ 4x - 2x^2 + x^3/3 \right]_1^2 = 2 + 16 - 12 - 2 = 4$$

$$\rightsquigarrow \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$\rightsquigarrow$ because $\delta_{i3}$ = 3rd row is 1, other 0.

$\delta_{j1}$ = 1st column is 1, other 0.

etc...

$$F_1 = \int_0^1 2(1-x)\,dx + \int_1^2 2(0)\,dx = 2\left[x - x^2/2\right]_0^1 = 1$$
$$F_2 = \int_0^1 2(x)\,dx + \int_1^2 2(2-x)\,dx = 2\left[x^2/2\right]_0^1 + \left[4x - x^2\right]_1^2 = 1 + 4 - 3 = 2$$
$$F_3 = F_1 = 1 \quad , \quad F = [1, 2, 1]^T$$

computed matrices:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

equation: $(A + B + C) \cdot u = F$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad \text{linear system}$$

- include BCs:

$u_1 = 1, \quad u_3 = 2$

use elimination approach:

$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} u_2 + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} 2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{1} \\ 2 \\ \cancel{1} \end{pmatrix} u_2 = \begin{pmatrix} \cancel{1} \\ 2 \\ \cancel{1} \end{pmatrix} - \begin{pmatrix} \cancel{2} \\ 2 \\ \cancel{0} \end{pmatrix} - \begin{pmatrix} \cancel{0} \\ 4 \\ \cancel{4} \end{pmatrix} = \begin{pmatrix} \cancel{-1} \\ -4 \\ \cancel{-3} \end{pmatrix}$$

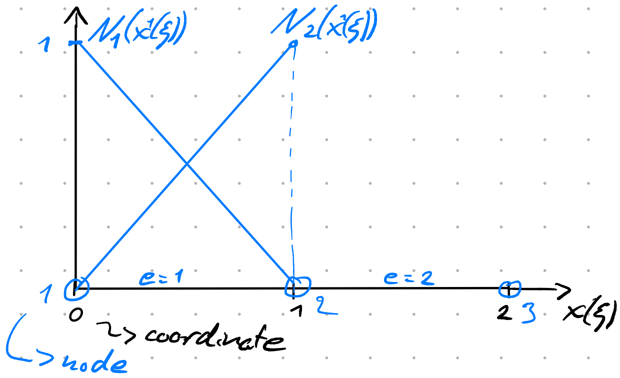
$$\rightsquigarrow \quad 2u_2 = -4$$
$$\Rightarrow u_2 = -2$$

Note: the <u>boundary term</u> does not need to be set up because only eq. 2 is of interest. The boundary equations need to be eliminated in the elimination approach.

## Problem 3

- use linear reference elements

$$\sum_j \Big[ \underbrace{\int_0^2 -N_i' N_j' \, dx}_{A_{ij}} + \underbrace{\int_0^2 6 N_i N_j \, dx}_{B_{ij}} \Big] u_j = \underbrace{\int_0^2 2 N_i \, dx}_{F_i} \quad \forall i$$




Element shape functions:

$$\begin{cases} N_1(\xi) = 1-\xi \\ N_2(\xi) = \xi \end{cases}, \quad \text{where } \xi \in [0, 1]$$

$A_{ij} = \sum_e A_{ij}^e$ , $i, j \in [1 \ldots \text{number of nodes}]$

-> reduced element matrices / vectors: $A_{ab}^e$ , $a, b \in [1 \ldots \text{number of element nodes}]$

-> assemble $A_{ab}^e$ into $A_{ij}$

Compute red. elem. matrices: $J_e = \frac{\partial x(\xi)}{\partial \xi} = h_e = 1$

$$A_{ab}^e = \int_0^1 (-1) N_a' N_b' \, \cancel{\det J_e}^{\,1} \, d\xi \overset{a=b}{=} \int_0^1 (-1)\, 1 \, d\xi = [\,\xi\,]_0^1 = -1$$

$$\overset{a \neq b}{=} \int_0^1 (-1)(1)(-1) \, d\xi = +1$$

$$[A^e] = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$[A] = \begin{bmatrix} -1 & +1 & 0 \\ +1 & -2 & +1 \\ 0 & +1 & -1 \end{bmatrix}$$

$$B_{ab}^e = \int_0^1 6 N_a N_b \, \cancel{\det J_e}^{\,1} \, d\xi \overset{a=b}{=} \int_0^1 6 \xi^2 \, d\xi = \frac{6}{3} [\,\xi^3\,]_0^1 = 2$$

$$\overset{a \neq b}{=} \int_0^1 6 \xi (1-\xi) \, d\xi = [\,3\xi^2 - 2\xi^3\,]_0^1 \; 1$$

$$[B^e] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$F_a^e = \int_0^1 2 N_a \, \cancel{\det J_e}^{\,1} \, d\xi \overset{a=1}{=} \int_0^1 2(1-\xi) \, d\xi = [\,2\xi - \xi^2\,]_0^1 = 1$$

$$\overset{b=2}{=} \int_0^1 2 \xi \, d\xi = [\,\xi^2\,]_0^1 = 1$$

$$[F^e] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[F] = [1, \; 2, \; 1]^T$$

- total eq.: $(\underline{\underline{A}} + \underline{\underline{B}}) \cdot \underline{u} = \underline{F}$

- We need only eq. for node #2 (node #1 and #3 are DBCs):

$$\left[ (1 \quad -2 \quad 1) + (1 \quad 4 \quad 1) \right] \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2$$

with $u_1 = 1$, $u_3 = 2$

$$\left[ 2 \quad 2 \quad 2 \right] \cdot \begin{pmatrix} 1 \\ u_2 \\ 2 \end{pmatrix} = 2$$

$\Rightarrow$ solve for $u_2$:

$$2 + 2u_2 + 4 = 2$$

$$\Rightarrow u_2 = -2$$

# Overview of class

1. Intro
   - What is CAE, FE?
   - Field theory
   - PDE + BCs $\hat{=}$ BVP

2. FE Method:

$$\boxed{\text{BVP}} \xrightarrow[\text{variation}]{} \boxed{\text{weak form}} \xrightarrow[\text{discretization}]{} \boxed{\text{algebraic system}} \xrightarrow[\text{numerical methods}]{} \boxed{\text{solution}}$$

- weak form: idea, meaning and derivation
- Galerkin discretization
  - -> mesh (nodes, elements, ansatz functions)
  - -> derivation of linear system
  - -> element matrices / vectors
  - -> assembling procedure

3. Electrostatics
   - derivation from Maxwell equations
   - electrostatic potential
   - physical meaning of BCs
   - when to use electrostatics

4. Mechanical field
   - field quantities: displacements, strain, stress
   - material law
   - Voigt notation
   - equations of motion
   - free harmonic motion (modal analysis)
   - meaning of BCs
   - 2D approximations: plain strain, plain stress
   - axi-symmetry
     -> curvi-linear coordinate system!

5. Magnetic Field
   - derivation from Maxwell eqs.
   - vector potential and gauge
   - 2D formulations (vector pot. normal to domain -> gauge)
   - meaning of BCs

6. FE method: handling irregular meshes
- reference elements
- isoparametric domain mapping
  - -> Jacobian matrix
- numerical integration to obtain element matrices / vectors

7. Acoustic field
- derivation of wave equation
- acoustic scalar potential
  - -> irrotational field (source-free regions)
  - -> only longitudinal waves
- BCs: sound soft, sound hard, impedance BC
- reflection coefficiend
  - -> impedance
- radiation problems / free-field simulations
- Helmholtz equation