Radiation of leaky Lamb waves: relation between attenuation and power flux

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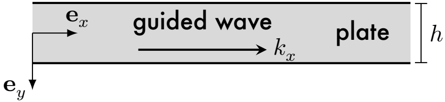
12th April 2022 – Marseille, France

Lamb waves



cross-section of a plate

- harmonic plane wave ansatz for the particle **displacements**:

$$\mathbf{u}(x,y,t) = \mathbf{u}(y)\,\mathrm{e}^{\mathrm{i}(k_x x - 2\pi f t)}$$

→ f: frequency, k_x: wavenumber

- eigenvalue problem for $\mathbf{u}(y)$, k_x:

Dispersion

$k_x = k_x(f)$ nonlinear

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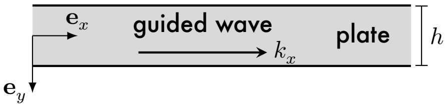

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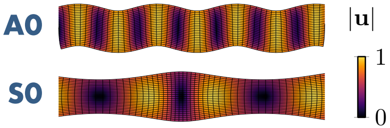
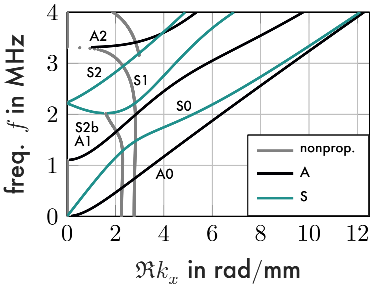
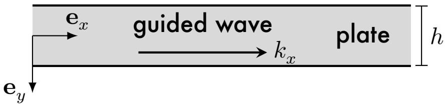


A0: anti-symmetric, **S0**: symmetric



brass plate, 1 mm

Lamb waves



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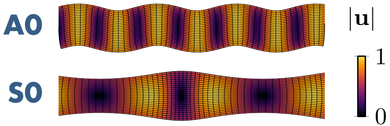

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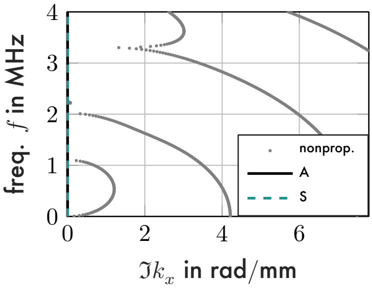

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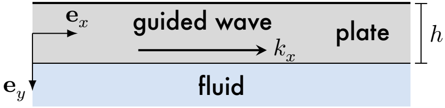

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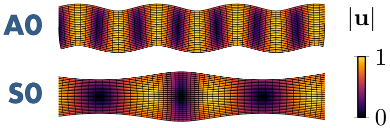

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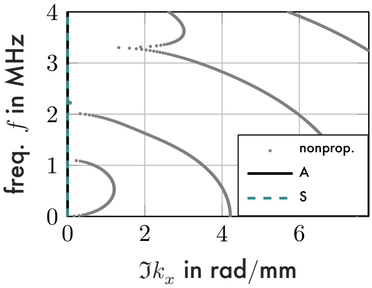

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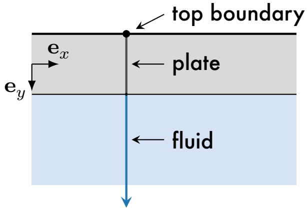
Models for a plate-fluid system

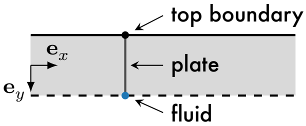
Different models to resolve $\mathbf{u}(y)$:

Full model:



- continuous spectrum
  → integral as solution

Open plate model:



- discrete spectrum
- plate-fluid resonances

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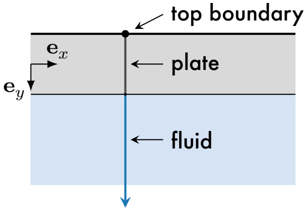

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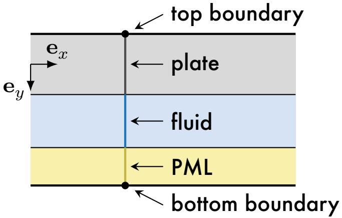
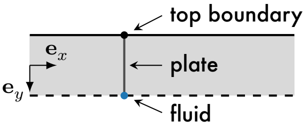
Truncated model:



- discrete spectrum
- some parameters unrelated to the physics

Open plate model:



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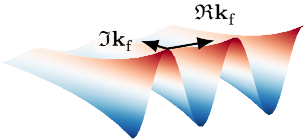
Contents

Quasi-guided waves

a-priori:

inhomogeneous plane wave in fluid



- complex wave vector: $\mathbf{k}_\mathrm{f} = \begin{bmatrix} k_x \\ k_y \end{bmatrix}$
- dispersion relation:

$$\mathbf{k}_\mathrm{f} \cdot \mathbf{k}_\mathrm{f} = \kappa_\mathrm{f}^2 = \frac{\omega^2}{c_\mathrm{f}^2} \in \mathbb{R} \quad \Rightarrow \quad \mathfrak{R}\mathbf{k}_\mathrm{f} \perp \mathfrak{I}\mathbf{k}_\mathrm{f}$$

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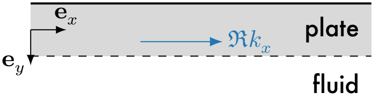

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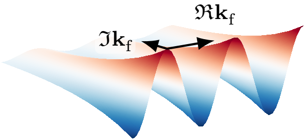

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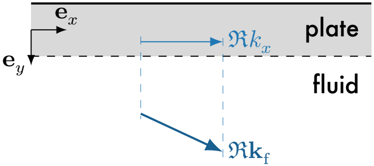

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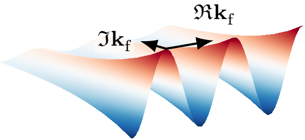

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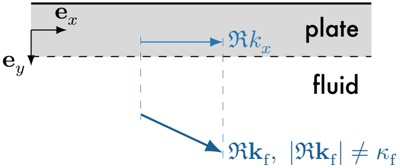

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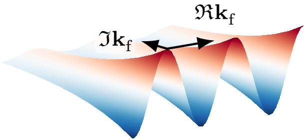

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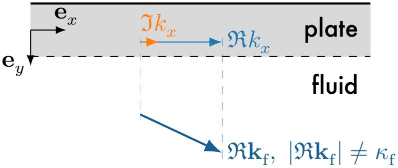

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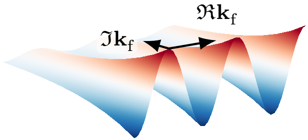

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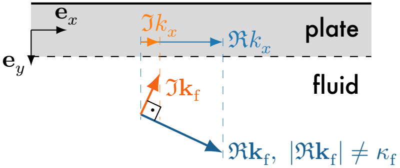

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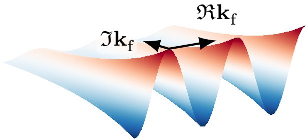

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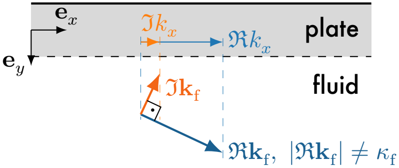
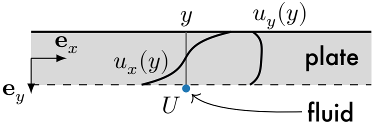


- displacement field:



→ additional unknown: **amplitude** U

Solving the quasi-guided wave problem[1]

⇒ nonlinear eigenvalue problem (EVP)

→ involving $k_y = \sqrt{\kappa_\mathrm{f}^2 - k_x^2}$

eigenvalue!

square root EVP in k_x

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[1] D. A. Kiefer et al. "Calculating the full leaky Lamb wave spectrum with exact fluid interaction." In: *The Journal of the Acoustical Society of America* 145.6 (June 2019), pp. 3341–3350.

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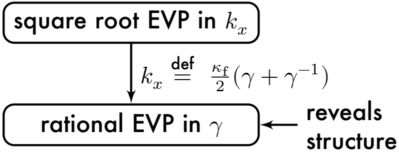

square root EVP in k_x

$k_x \overset{\text{def}}{=} \frac{\kappa_\mathrm{f}}{2}(\gamma + \gamma^{-1})$

rational EVP in γ

reveals structure

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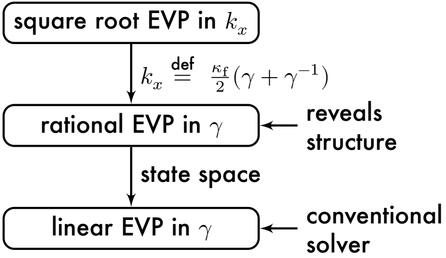

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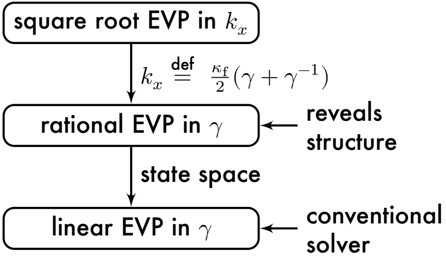

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rational EVP in γ ← reveals structure

state space

linear EVP in γ ← conventional solver

- ✓ reliable and efficient
- ✓ exact fluid-structure interaction
- ✓ uniquely obtain $[k_x, k_y]$

- dispersion curves (200 freq.): approx. $1\,$s – $10\,$s

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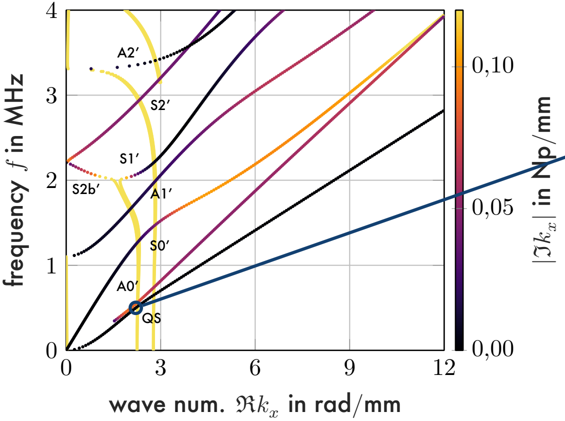
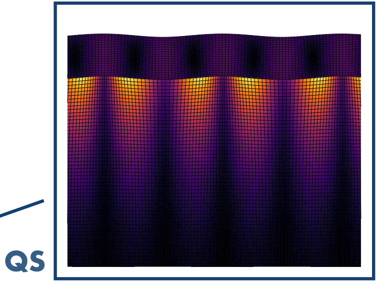
Trapped and leaky wave solutions





- trapped wave
- transport of energy along the plate

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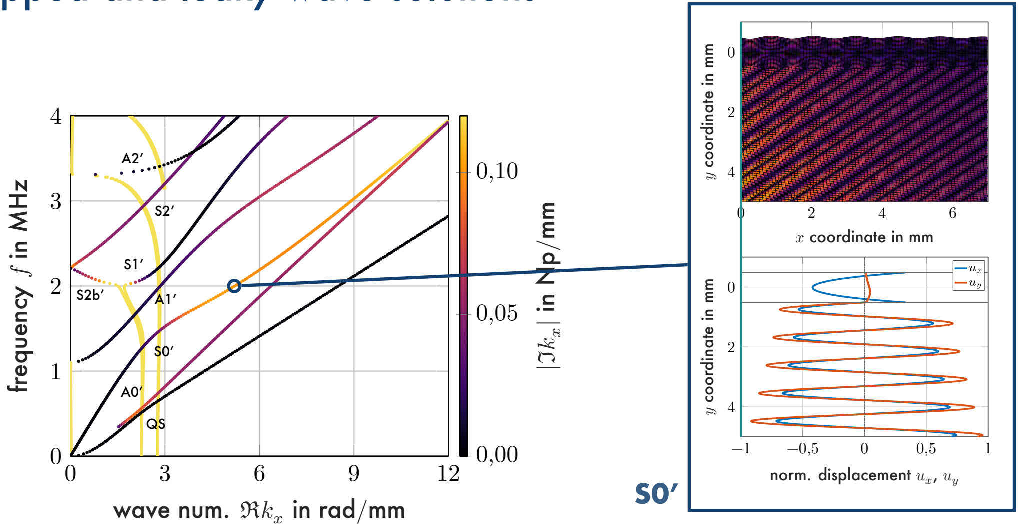

- leaky wave

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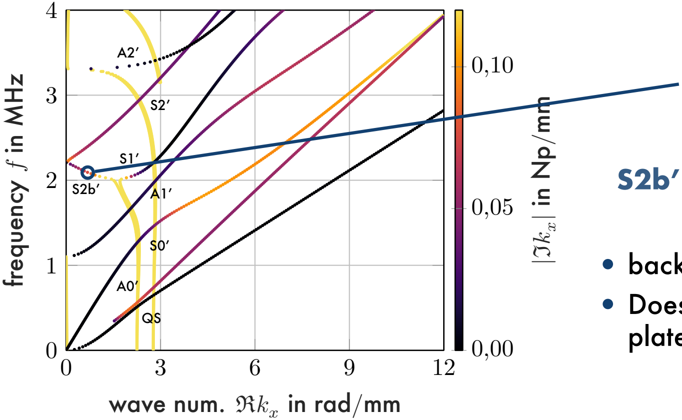
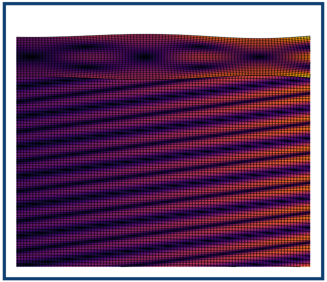
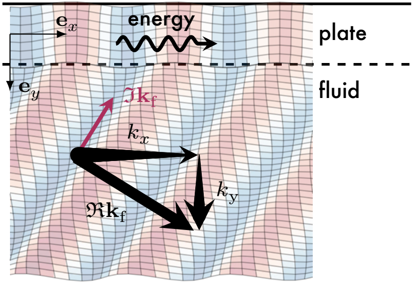




- backward leaky wave
- Does not diverge with distance to the plate.

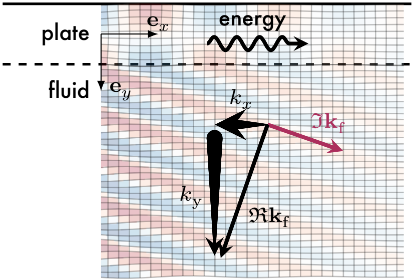
Radiation of forward vs. backward waves

forward wave:



diverges with distance from plate

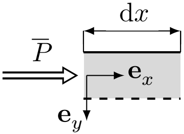
backward wave:



confined to proximity of plate

Radiation rate: leakage

- due to radiation: $\mathbf{u} \sim \mathrm{e}^{-\alpha x}$
  - $\rightarrow$ α : radiation rate
    (attenuation due to leakage <u>only</u>)



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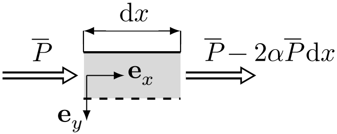

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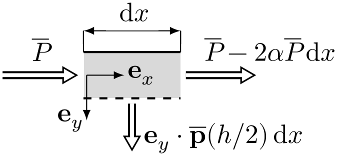

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$\Rightarrow$ radiation rate: $\alpha = \dfrac{\mathbf{e}_y \cdot \overline{\mathbf{p}}(h/2)}{2\overline{P}}$

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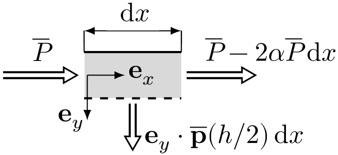

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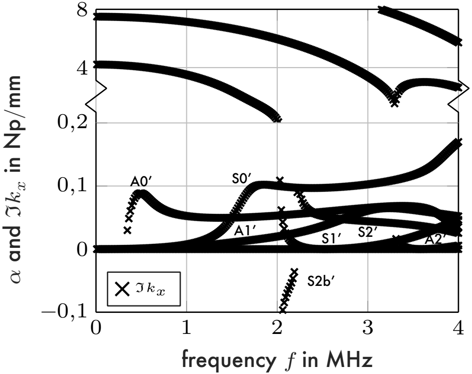

radiation rate vs. attenuation: brass-water

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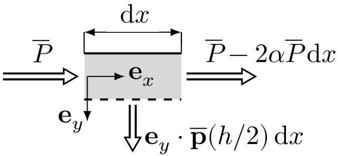

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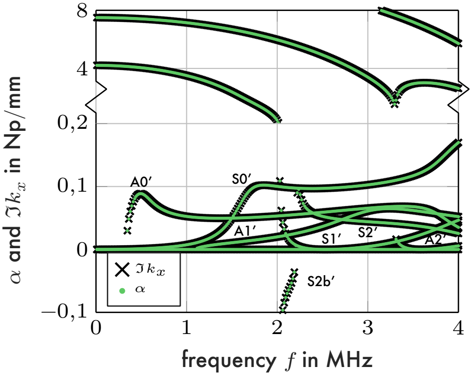

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$\Rightarrow$ $\alpha = \Im k_x$

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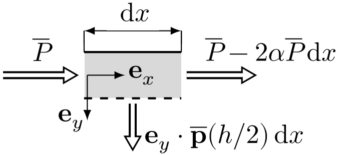

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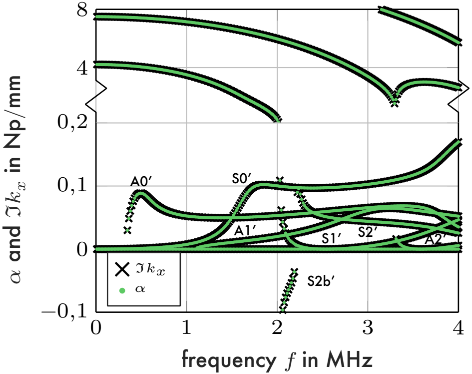
$\Rightarrow$

$$\text{radiation rate: } \alpha = \frac{\mathbf{e}_y \cdot \overline{\mathbf{p}}(h/2)}{2\overline{P}}$$



radiation rate vs. attenuation: brass-water

$\Rightarrow$

$$\alpha(\underset{\uparrow}{\mathbf{u}}) = \Im k_x$$

eigenvectors

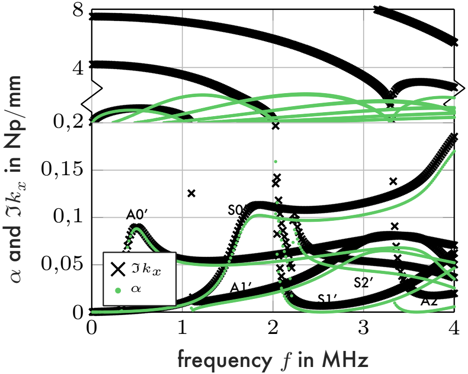
Dissipative plate

- **Young's modulus:** $E = E_0(1 - \mathrm{i}0.0025)$

$$\Rightarrow \boxed{\alpha \neq \Im k_x}$$

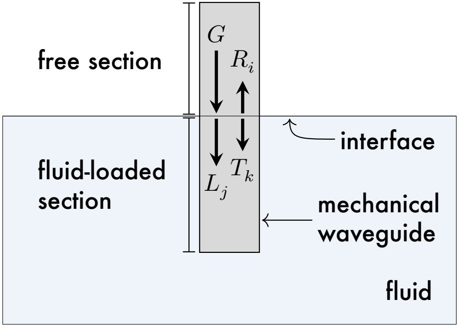
- **wave field:** $\mathbf{u} \sim \mathrm{e}^{-\alpha x}\,\mathrm{e}^{-dx}$

α characterizes the radiation process

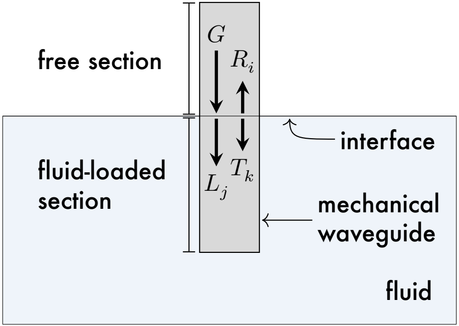


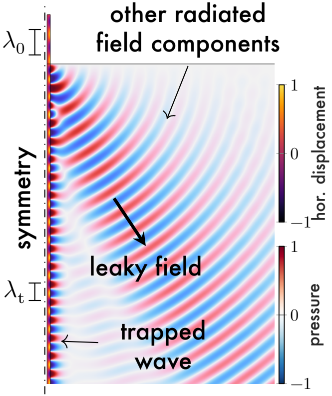
radiation rate α vs. attenuation $\Im k_x$ of a dissipative plate

Immersed plate: mode conversion



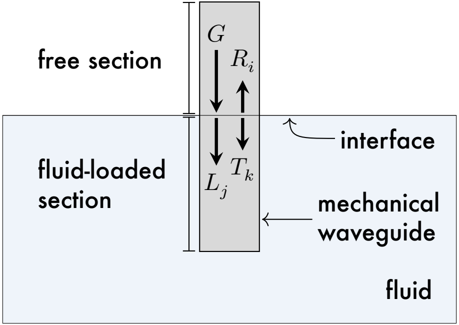
Immersed plate: mode conversion

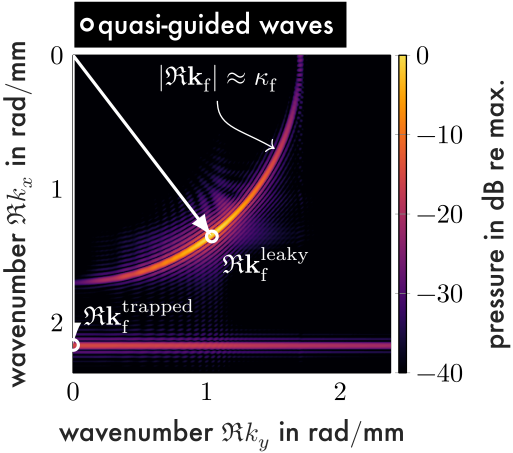




Finite Element computation: PMMA-water at 400 kHz, thickness of 1 mm

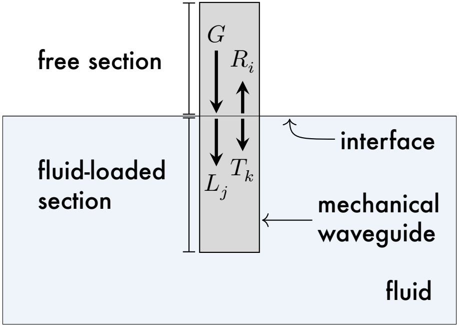
Immersed plate: mode conversion





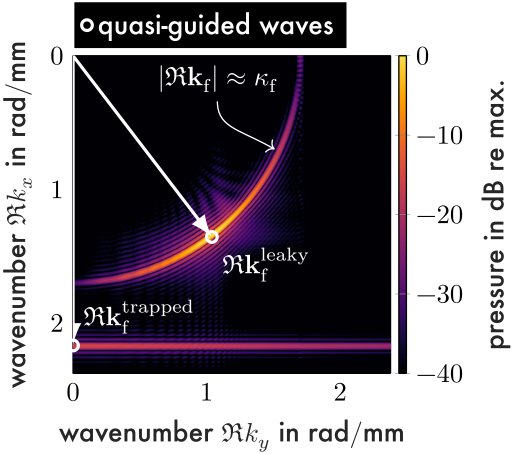
spectrum in water at 400 kHz

Immersed plate: mode conversion



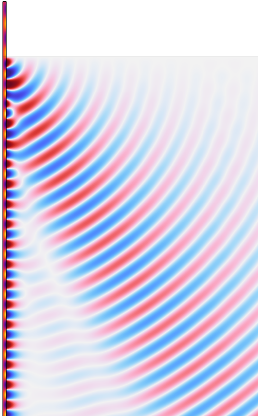
But:

How to compute transmission/reflection?



spectrum in water at 400 kHz

Conclusion and outlook

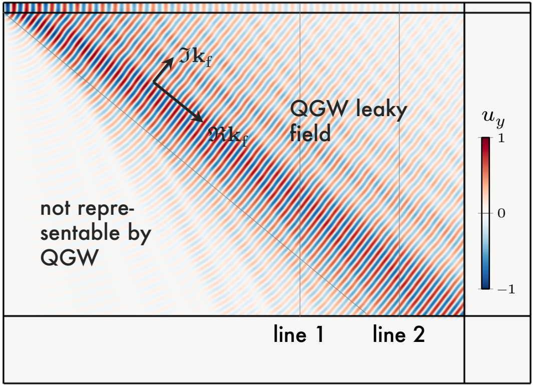


- Analysis of
  - → power flux and
  - → wave field

  of quasi-guided waves
- Design of sensors
  - → "principal field components" are desired
  - → simple and accurate model
- Development of a

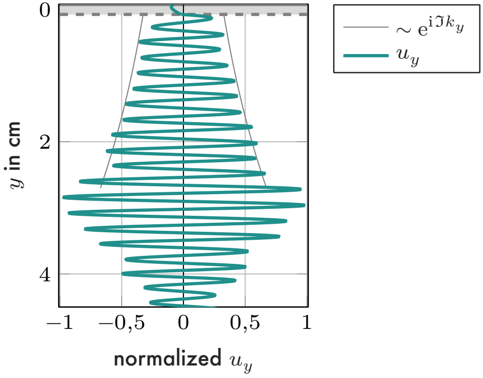
quasinormal mode theory

FE computation: leaky field

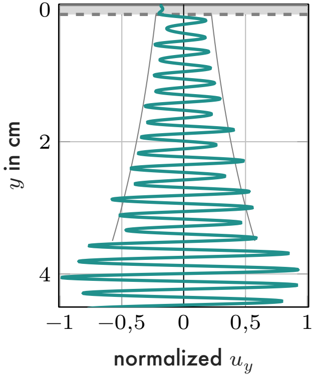


brass plate and water: A0′ wave is excited at 1.5 MHz mm

FE computation: diverging wave field

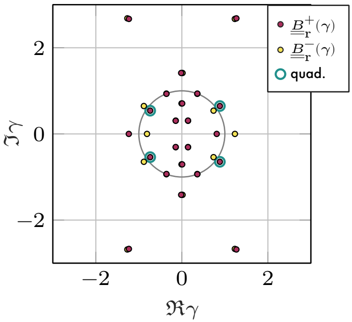


4.5 cm from source

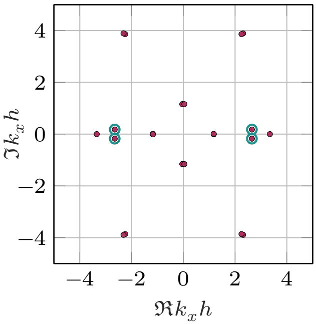


6 cm from source

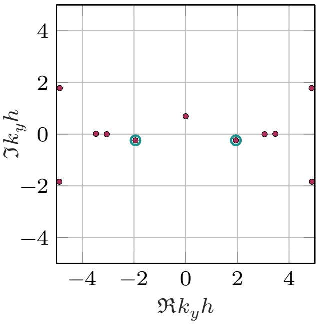
Symmetry of spectrum



eigenvalues

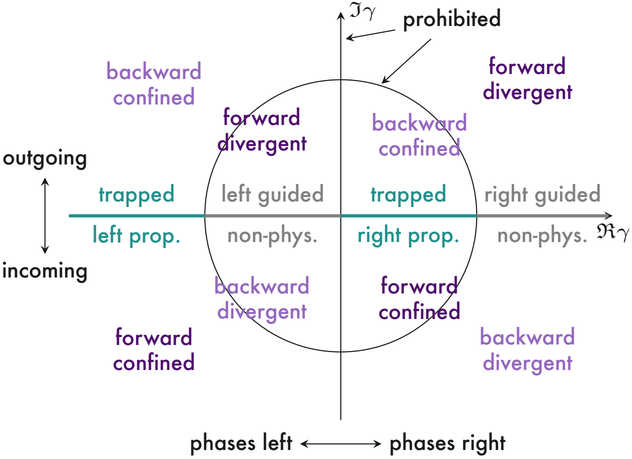


axial wavenumbers



transversal wavenumbers

Classification in γ-plane



$\Im\gamma$

prohibited

backward confined

forward divergent

forward divergent

backward confined

outgoing

trapped

left guided

trapped

right guided

left prop.

non-phys.

right prop.

non-phys.

$\Re\gamma$

incoming

backward divergent

forward confined

forward confined

backward divergent

phases left ⟷ phases right

■ forward ■ backward ■ trapped

waves fully segregate into 12 categories